

Nonlinear Dirac solitons in external fields

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We consider the nonlinear Dirac equations (NLDE's) in 1+1 dimension with scalar-scalar self interaction $\frac{g^2}{\kappa+1}(\bar{\Psi}\Psi)^{\kappa+1}$ in the presence of various external electromagnetic fields. We find exact solutions for special external fields and we study the behavior of solitary wave solutions to the NLDE in the presence of a wide variety of fields in a variational approximation depending on collective coordinates which allows the position, width and phase of these waves to vary in time. We find that in this approximation the position $q(t)$ of the center of the solitary wave obeys the usual behavior of a relativistic point particle in an external field. For time independent external fields we find that the energy of the solitary wave is conserved but not the momentum which becomes a function of time. We postulate that similar to the nonlinear Schrödinger equation (NLSE) that a sufficient dynamical condition for instability to arise is that $dP(t)/d\dot{q}(t) < 0$. Here $P(t)$ is the momentum of the solitary wave, and \dot{q} is the velocity of the center of the wave. We investigate the accuracy of our variational approximation using numerical simulations of the NLDE and find that when the forcing term is small and we are in a regime where the solitary wave is stable, that the behavior of the solutions of the collective coordinate equations agrees very well with the numerical simulations.

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I. INTRODUCTION

Classical solutions of nonlinear field equations have a long history as a model of extended particles [1, 2]. In 1970, Soler [2] proposed that the self-interacting 4-Fermi theory was an interesting model for extended fermions. Later, Strauss and Vasquez [3] were able to study the stability of this model under dilatation and found the domain of stability for the Soler solutions. Solitary waves in the 1+1 dimensional nonlinear Dirac equation (NLDE) have been studied [4, 5] in the past in the case of massive Gross-Neveu [6] (with $N = 1$, i.e. just one localized fermion) and massive Thirring [7] models). In those studies it was found that these equations have solitary wave solutions for both scalar-scalar (S-S) and vector-vector (V-V) interactions. The interaction between solitary waves of different initial charge was studied in detail [8] for the S-S case in the work of Alvarez and Carreras by Lorentz boosting the static solutions and allowing them to scatter. Recently we extended the solutions previously found to a more general interaction of the form $\frac{g^2}{\kappa+1}(\bar{\Psi}\Psi)^{\kappa+1}$ [9]. For the nonrelativistic limit of the NLDE, namely the nonlinear Schrödinger equation (NLSE), there have been recent studies of the behavior of the NLSE in external fields. Using a collective coordinate theory, the authors found [10] [11], [12] [13] that a sufficient dynamical condition for instability to arise is that $dp(t)/dv < 0$. Here $p(t)$ is the normalized canonical momentum $p(t) = \frac{1}{M(t)} \frac{\partial L}{\partial \dot{q}}$, $M(t) = \int dx \Psi^*(x, t) \Psi(x, t)$ is the mass and $\dot{q}(t) = v(t)$ is the velocity of the solitary wave.

One of the points we will investigate in the paper is whether this dynamical stability criterion is also valid for the NLDE. There has been recent interest in the stability of NLDE with higher-order nonlinearity [14]. Comech (private communication) has been able to prove that for $\kappa = 1$, the Vakhitov-Kolokolov [15] criterion guarantees linear stability in the non-relativistic regime of the NLDE equation for solutions of the form (in the rest frame) $\Psi(x, t) = \psi(x)e^{-i\omega t}$ where ω is less than but approximately equal to the mass parameter m in the Dirac equation. He was also able to show linear instability in the same nonrelativistic regime for $\kappa \geq 3$. This is the first rigorous result for the Dirac

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equation, but it only applies in the nonrelativistic regime. Here we want to understand if we can determine in the relativistic regime for what values of ω do the solitary waves become unstable, with and without forcing terms even when they are stable in the non-relativistic regime.

This paper is organized as follows: In Sec. II, we review the known exact solutions and discuss the conservation laws that govern their behavior. In Sec. III we extend Bogolubsky's discussion [16] of the stability of solitary waves to arbitrary nonlinearity parameter κ . In Sec. IV we consider the NLDE in external electromagnetic fields, find particular exact solitary wave solutions, and discuss the stability of these solutions. In Sec. V we introduce our variational method based on using the exact solution wave functions for the unforced problem, while having the position, width parameter and phase become collective coordinates depending on time. We write the relativistic equations for these collective coordinates which are similar to point particle relativistic dynamical equations. In Sec. VI we postulate our stability criteria for an arbitrary external potential. In Sec. VII we examine and solve the collective coordinate (CC) equations for three types of potentials—a ramp potential, a harmonic potential and a spatially periodic potential. We also compare the solution to the CC equations to the numerical simulation of the NLDE equation. In the appendix we discuss identities that are obeyed by the solutions in the rest frame.

II. REVIEW OF EXACT SOLUTIONS TO THE NLDE

In this section we review the exact solutions to the NLDE, using the notation of [9]. We are interested in solitary wave solution of the NLDE given by

$$(i\gamma^\mu \partial_\mu - m)\Psi + g^2(\bar{\Psi}\Psi)^\kappa \Psi = 0. \quad (2.1)$$

These equations can be derived in a standard fashion from the Lagrangian

$$L = \left(\frac{i}{2}\right) [\bar{\Psi}\gamma^\mu \partial_\mu \Psi - \partial_\mu \bar{\Psi}\gamma^\mu \Psi] - m\bar{\Psi}\Psi + \frac{g^2}{\kappa + 1} (\bar{\Psi}\Psi)^{\kappa+1}. \quad (2.2)$$

For solitary wave solutions, the field Ψ goes to zero at infinity. It is sufficient to go into the rest frame, since the theory is Lorentz invariant and the moving solution can be obtained by a Lorentz boost. In the rest frame we consider solutions of the form

$$\Psi(x, t) = e^{-i\omega t} \psi(x). \quad (2.3)$$

We are interested in bound state solutions that correspond to positive energy $\omega \geq 0$ and which have energies in the rest frame less than the mass parameter m , i.e. $\omega < m$. In our previous paper [9], we chose the representation $\gamma_0 = \sigma_3$, $i\gamma_1 = \sigma_1$. Here instead, to make contact with the numerical simulations paper of Alvarez and Carreras [8] we choose instead

$$\gamma^0 = \sigma_3; \quad \gamma^1 = i\sigma_2. \quad (2.4)$$

Defining A, B via:

$$\psi(x) = \begin{pmatrix} A(x) \\ i B(x) \end{pmatrix} = R(x) \begin{pmatrix} \cos \theta \\ i \sin \theta \end{pmatrix}, \quad (2.5)$$

we obtain the following equations for A and B .

$$\begin{aligned} \frac{dA}{dx} + (m + \omega)B - g^2(A^2 - B^2)^\kappa B &= 0, \\ \frac{dB}{dx} + (m - \omega)A - g^2(A^2 - B^2)^\kappa A &= 0. \end{aligned} \quad (2.6)$$

A first integral of these equations can be obtained by realizing that from energy-momentum conservation we have

$$\partial_\mu T^{\mu\nu} = 0; \quad T^{\mu\nu} = \frac{i}{2} [\bar{\Psi}\gamma^\mu \partial^\nu \Psi - \partial^\nu \bar{\Psi}\gamma^\mu \Psi] - g^{\mu\nu} L. \quad (2.7)$$

Thus for stationary solutions

$$T^{10} = \text{const.} \quad T^{11} = \text{const.} \quad (2.8)$$

Now using (2.3) we obtain

$$T^{11} = \omega\psi^\dagger\psi - m\bar{\psi}\psi + L_I, \quad (2.9)$$

where

$$L_I = \frac{g^2}{\kappa + 1}(\bar{\psi}\psi)^{\kappa+1}. \quad (2.10)$$

For solitary wave solutions vanishing at infinity the constant is zero and we get the useful first integral:

$$T^{11} = \omega\psi^\dagger\psi - m\bar{\psi}\psi + L_I = 0. \quad (2.11)$$

Multiplying the equation of motion on the left by $\bar{\Psi}$ and using (2.3) we have that:

$$(\kappa + 1)L_I = -\omega\psi^\dagger\psi + m\bar{\psi}\psi - \bar{\psi}i\gamma^1\partial_1\psi. \quad (2.12)$$

Therefore we can rewrite $T^{11} = 0$ as

$$\omega\kappa\psi^\dagger\psi - m\kappa\bar{\psi}\psi - \bar{\psi}i\gamma^1\partial_1\psi = 0. \quad (2.13)$$

For the Hamiltonian density we obtain

$$\mathcal{H} = T^{00} = \frac{i}{2} [\bar{\Psi}\gamma^1\partial_x\Psi - \partial_x\bar{\Psi}\gamma^1\Psi] + m\bar{\Psi}\Psi - L_I \equiv h_1 + h_2 - h_3. \quad (2.14)$$

Each of the h_i are positive definite. From Eq. (2.11) and (2.12) one has the relationship:

$$\kappa L_I = -\bar{\psi}i\gamma^1\partial_x\psi. \quad (2.15)$$

From this we have

$$h_3 = \frac{1}{\kappa}h_1 \quad (2.16)$$

and in particular for $\kappa = 1$, $\mathcal{H} = m\bar{\psi}\psi$.

In terms of R, θ one has

$$\bar{\psi}i\gamma_1\partial_1\psi = \psi^\dagger\psi \frac{d\theta}{dx}. \quad (2.17)$$

This leads to the simple differential equation for θ for solitary waves

$$\frac{d\theta}{dx} = -\omega_\kappa + m_\kappa \cos 2\theta; \quad \text{where } \omega_\kappa \equiv \kappa \omega; \quad m_\kappa = \kappa m. \quad (2.18)$$

The solution is

$$\theta(x) = \tan^{-1}(\alpha \tanh \beta_\kappa x), \quad (2.19)$$

where

$$\alpha = \left(\frac{m_\kappa - \omega_\kappa}{m_\kappa + \omega_\kappa} \right)^{1/2} = \left(\frac{m - \omega}{m + \omega} \right)^{1/2}, \quad \beta_\kappa = (m_\kappa^2 - \omega_\kappa^2)^{1/2}. \quad (2.20)$$

Thus we have

$$\begin{aligned} \tan \theta(x) &= \alpha \tanh \beta_\kappa x, \\ \sin^2 \theta(x) &= \frac{\alpha^2 \tanh^2 \beta_\kappa x}{1 + \alpha^2 \tanh^2 \beta_\kappa x} = \frac{(m - \omega) \sinh^2 \beta_\kappa x}{m \cosh 2\beta_\kappa x + \omega}, \\ \cos^2 \theta(x) &= \frac{1}{1 + \alpha^2 \tanh^2 \beta_\kappa x} = \frac{(m + \omega) \cosh^2 \beta_\kappa x}{m \cosh 2\beta_\kappa x + \omega}, \end{aligned} \quad (2.21)$$

where we have used the identities:

$$\begin{aligned} 1 + \alpha^2 \tanh^2 \beta_k x &= \left(\frac{m \cosh 2\beta_k x + \omega}{m + \omega} \right) \operatorname{sech}^2 \beta_k x, \\ 1 - \alpha^2 \tanh^2 \beta_k x &= \left(\frac{\omega \cosh 2\beta_k x + m}{m + \omega} \right) \operatorname{sech}^2 \beta_k x. \end{aligned} \quad (2.22)$$

We have from Eq. (2.10) and Eq. (2.11)

$$\omega R^2 - m R^2 \cos 2\theta + \frac{g^2}{\kappa + 1} (R^2 \cos 2\theta)^{\kappa+1} = 0. \quad (2.23)$$

Thus

$$R^2 = \left[\frac{(\kappa + 1)(m \cos 2\theta - \omega)}{g^2 (\cos 2\theta)^{\kappa+1}} \right]^{1/\kappa}. \quad (2.24)$$

Now we have

$$\frac{d\theta}{dx} = \frac{\beta_\kappa^2}{\omega_\kappa + m_\kappa \cosh 2\beta_\kappa x} = -\omega_\kappa + m_\kappa \cos 2\theta, \quad (2.25)$$

where $\beta_\kappa = \sqrt{m_\kappa^2 - \omega_\kappa^2} = \kappa \sqrt{m^2 - \omega^2}$, so that

$$\cos 2\theta = \frac{m_\kappa + \omega_\kappa \cosh 2\beta_\kappa x}{\omega_\kappa + m_\kappa \cosh 2\beta_\kappa x} = \frac{m + \omega \cosh 2\beta_\kappa x}{\omega + m \cosh 2\beta_\kappa x}. \quad (2.26)$$

One important expression is

$$m \cos 2\theta - \omega = \frac{\beta_\kappa^2}{\kappa^2 (\omega + m \cosh 2\beta_\kappa x)}. \quad (2.27)$$

Using this we get

$$R^2 = \left(\frac{\omega + m \cosh 2\beta_\kappa x}{m + \omega \cosh 2\beta_\kappa x} \right) \left[\frac{(\kappa + 1)\beta_\kappa^2}{g^2 \kappa^2 (m + \omega \cosh 2\beta_\kappa x)} \right]^{1/\kappa}. \quad (2.28)$$

Using the identities in Eq. (2.22), we obtain the alternate expression

$$R^2 = \left(\frac{1 + \alpha^2 \tanh^2 \beta_\kappa x}{1 - \alpha^2 \tanh^2 \beta_\kappa x} \right) \left[\frac{\operatorname{sech}^2 \beta_\kappa x (\kappa + 1) \beta_\kappa^2}{g^2 \kappa^2 (m + \omega) (1 - \alpha^2 \tanh^2 \beta_\kappa x)} \right]^{1/\kappa}. \quad (2.29)$$

In particular for $\kappa = 1$

$$R^2 = \frac{2(m - \omega)}{g^2} \frac{(1 + \alpha^2 \tanh^2 \beta x)}{(1 - \alpha^2 \tanh^2 \beta x)^2} \operatorname{sech}^2 \beta x \quad (2.30)$$

and

$$\begin{aligned} A^2 &= R^2 \cos^2 \theta = \frac{2}{g^2} \frac{(m^2 - \omega^2)(m + \omega) \cosh^2 \beta x}{(m + \omega \cosh 2\beta x)^2}, \\ B^2 &= R^2 \sin^2 \theta = \frac{2}{g^2} \frac{(m^2 - \omega^2)(m - \omega) \sinh^2 \beta x}{(m + \omega \cosh 2\beta x)^2}. \end{aligned} \quad (2.31)$$

We can also write

$$\cos \theta = \sqrt{\frac{(m + \omega) \cosh^2(\kappa \beta x)}{\omega + m \cosh(2\kappa \beta x)}}, \quad \sin \theta = \sqrt{\frac{(m - \omega) \sinh^2(\kappa \beta x)}{\omega + m \cosh(2\kappa \beta x)}}, \quad (2.32)$$

so that

$$\begin{aligned} A &= \sqrt{\frac{(m+\omega)\cosh^2(\kappa\beta x)}{m+\omega\cosh(2\kappa\beta x)}} \left[\frac{(\kappa+1)\beta^2}{g^2(m+\omega\cosh(2\kappa\beta x))} \right]^{\frac{1}{2\kappa}}, \\ B &= \sqrt{\frac{(m-\omega)\sinh^2(\kappa\beta x)}{m+\omega\cosh(2\kappa\beta x)}} \left[\frac{(\kappa+1)\beta^2}{g^2(m+\omega\cosh(2\kappa\beta x))} \right]^{\frac{1}{2\kappa}}. \end{aligned} \quad (2.33)$$

Because of Lorentz invariance we can find the solution in a frame moving with velocity v with respect to the rest frame. The Lorentz boost is given in terms of the rapidity variable η as follows (here $c = 1$):

$$v = \tanh \eta; \quad \gamma = \frac{1}{\sqrt{1-v^2}} = \cosh \eta; \quad \sinh \eta = \frac{v}{\sqrt{1-v^2}}. \quad (2.34)$$

In the moving frame, the transformation law for spinors tells us that:

$$\Psi(x, t) = \begin{pmatrix} \cosh(\eta/2) & \sinh(\eta/2) \\ \sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix} \begin{pmatrix} \Psi_1^0[\gamma(x-vt), \gamma(t-vx)] \\ \Psi_2^0[\gamma(x-vt), \gamma(t-vx)] \end{pmatrix}, \quad (2.35)$$

since

$$\cosh(\eta/2) = \sqrt{(1+\gamma)/2}; \quad \sinh(\eta/2) = \sqrt{(\gamma-1)/2}, \quad (2.36)$$

This in component form:

$$\begin{aligned} \Psi_1(x, t) &= (\cosh(\eta/2)A(x') + i \sinh(\eta/2)B(x')) e^{-i\omega t'}, \\ \Psi_2(x, t) &= (\sinh(\eta/2)A(x') + i \cosh(\eta/2)B(x')) e^{-i\omega t'}, \end{aligned} \quad (2.37)$$

where

$$x' = \gamma(x - vt); \quad t' = \gamma(t - vx). \quad (2.38)$$

Note that $\cosh^2(\eta/2) + \sinh^2(\eta/2) = \cosh \eta = \gamma$.

A. Conservation Laws of the NLDE

The Lagrangian is invariant under the transformation of phase $\Psi \rightarrow e^{i\Lambda}\Psi$, which by Noether's theorem leads to the conserved current:

$$\partial_\mu j^\mu(x) = 0; \quad j^\mu = \bar{\Psi} \gamma^\mu \Psi. \quad (2.39)$$

This leads to charge conservation:

$$Q = \int dx \Psi^\dagger \Psi, \quad (2.40)$$

which for the solitary wave solution leads to

$$Q = \int dx (A^2 + B^2) = \frac{1}{\kappa\beta} \left[\frac{(\kappa+1)\beta^2}{g^2(m+\omega)} \right]^{1/\kappa} I_\kappa(\alpha^2), \quad (2.41)$$

where

$$\begin{aligned} I_\kappa(\alpha^2) &= \int_{-1}^1 dy \frac{1 + \alpha^2 y^2}{(1 - y^2)^{(\kappa-1)/\kappa} [1 - \alpha^2 y^2]^{(\kappa+1)/\kappa}} \\ &= B \left(\frac{1}{2}, \frac{1}{\kappa} \right) {}_2F_1 \left(1 + \frac{1}{\kappa}, \frac{1}{2}, \frac{1}{2} + \frac{1}{\kappa}; \alpha^2 \right) \\ &\quad + \alpha^2 B \left(\frac{3}{2}, \frac{1}{\kappa} \right) {}_2F_1 \left(1 + \frac{1}{\kappa}, \frac{3}{2}, \frac{3}{2} + \frac{1}{\kappa}; \alpha^2 \right), \end{aligned} \quad (2.42)$$

and $\alpha^2 = \frac{m-\omega}{m+\omega}$, ${}_2F_1$ is a hypergeometric function and $B(x, k)$ denotes the beta function.

We also have energy-momentum conservation Eq. (2.7) leading to conservation of energy:

$$E = \int T^{00} dx, \quad (2.43)$$

and momentum conservation

$$P = \int T^{01} dx. \quad (2.44)$$

Because of Lorentz invariance it is sufficient to calculate the energy momentum tensor in the comoving frame $v = 0$. The energy momentum tensor in an arbitrary frame is then given by

$$T^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta}, \quad (2.45)$$

where

$$\Lambda^\mu_\alpha = \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix}. \quad (2.46)$$

In the rest frame of the solitary wave, for the unperturbed system one has that

$$T_{00} = h_1 \left(1 - \frac{1}{\kappa} \right) + h_2, \quad (2.47)$$

where

$$h_1 = R^2(x) \frac{d\theta}{dx} = \frac{\kappa\beta^2}{m + \omega \cosh(2\kappa\beta x)} \left[\frac{(\kappa + 1)\beta^2}{g^2(m + \omega \cosh(2\kappa\beta x))} \right]^{1/\kappa}, \quad (2.48)$$

$$h_2 = m\bar{\psi}\psi = m(A^2 - B^2) = m \left[\frac{(\kappa + 1)\beta^2}{g^2(m + \omega \cosh(2\kappa\beta x))} \right]^{1/\kappa}. \quad (2.49)$$

Integrating in the rest frame, we get for the rest frame energy

$$E_0 = H_1 \left(1 - \frac{1}{\kappa} \right) + H_2, \quad (2.50)$$

where

$$\begin{aligned} H_1 &= \int dx h_1 = \frac{\beta}{m + \omega} \left[\frac{(\kappa + 1)\beta^2}{g^2(m + \omega)} \right]^{1/\kappa} \\ &\times B \left(\frac{1}{2}, 1 + \frac{1}{\kappa} \right) {}_2F_1 \left(1 + \frac{1}{\kappa}, \frac{1}{2}, \frac{3}{2} + \frac{1}{\kappa}; \alpha^2 \right), \end{aligned} \quad (2.51)$$

$$\begin{aligned} H_2 &= \int dx h_2 = \frac{1}{\kappa\beta} \left[\frac{(\kappa + 1)\beta^2}{g^2(m + \omega)} \right]^{1/\kappa} \\ &\times B \left(\frac{1}{2}, \frac{1}{\kappa} \right) {}_2F_1 \left(\frac{1}{\kappa}, \frac{1}{2}, \frac{1}{2} + \frac{1}{\kappa}; \alpha^2 \right). \end{aligned} \quad (2.52)$$

Since in the rest frame for stationary solutions $T^{11} = T^{01} = 0$, the energy of the solitary wave in the moving frame is just

$$E = E_0 \cosh \eta = \gamma E_0; \quad P = E_0 \sinh \eta, \quad (2.53)$$

so that the norm $E^2 - P^2 = E_0^2 = M_0^2$.

In particular, for $\kappa = 1$ and $m = 1$, we have that

$$M_0 = \frac{2}{g^2 Q} \sinh^{-1} \frac{g^2 Q}{2}, \quad (2.54)$$

with Q and ω related as follows:

$$Q = \frac{2\beta}{g^2 \omega}; \quad \beta = \sqrt{1 - \omega^2}. \quad (2.55)$$

We also have

$$\begin{aligned} H_1 &= -\frac{2 \left(\sqrt{1 - \omega^2} - 2 \tanh^{-1} \left(\sqrt{\frac{1 - \omega}{\omega + 1}} \right) \right)}{g^2}, \\ H_2 &= \frac{4 \tanh^{-1} \left(\sqrt{\frac{1 - \omega}{\omega + 1}} \right)}{g^2}. \end{aligned} \quad (2.56)$$

III. STABILITY OF EXACT SOLUTIONS

A. Stability to changes in the frequency at fixed charge

Bogolubsky [16] suggested that the stability could be ascertained by looking at variations of the wave function, keeping the charge fixed and seeing if the solution was a minimum (stable to that variation) or maximum (unstable to that variation) of the Hamiltonian as a function of the parameter ω . This principle has been very useful in the past to determining the stability of scalar wave equations that are Hamiltonian dynamical systems. If the variation decreases the energy it turned out that the solitons were unstable. Since in higher dimensions there are many degrees of freedom for perturbing the system, this criterion is a sufficient condition for instability. For the Dirac case we have found from our numerical simulations that this criterion does not determine the critical ω except when $\kappa = 1$ [17], the case originally studied by Bogolubsky [16]. Assuming we know the wave function at the value of ω corresponding to a fixed charge Q , if we change the parametric dependence on ω this also changes the charge. This can be corrected by assuming that the new wave function has a new normalization that corrects for this. That is if we parametrize a rest frame solitary wave solution of the NLDE which has a charge $Q[\omega]$ by

$$\psi_s(x, t) = \chi_s(x, \omega) e^{-i\omega t}, \quad (3.1)$$

then we choose our slightly changed wave function to be

$$\begin{aligned} \tilde{\psi}[x, t, \omega', \omega] &= \frac{\sqrt{Q[\omega]}}{\sqrt{Q[\omega']}} \chi_s(x, \omega') e^{-i\omega' t} \\ &\equiv f(\omega', \omega) \chi_s(x, \omega') e^{-i\omega' t}. \end{aligned} \quad (3.2)$$

Then the wave function $\tilde{\psi}[x, t, \omega', \omega]$ has the same charge as $\psi[x, t, \omega]$. Inserting this wave function into the Hamiltonian we get a new Hamiltonian H_p depending on both ω', ω . As a function of ω' this new Hamiltonian is stationary as a function of ω' at the value $\omega' = \omega$. The criterion Bogolubsky proposed [16] is that the solitary wave is stable (unstable) according to whether this new Hamiltonian has a minimum (maximum) at $\omega' = \omega$. What we will find for $\kappa = 1$ is that there is a critical value of ω (determined by the coupling g and Q) below which the solitary wave is unstable, and this result is borne out by numerical simulations which we will present below. However, we will present in another paper numerical simulations at arbitrary κ which suggest that this method gives a sufficient condition for instability [17]. The probe Hamiltonian has the form:

$$H_p[\omega', \omega] = H_1[\omega'] \left(f(\omega', \omega)^2 - \frac{1}{\kappa} f(\omega', \omega)^{2(\kappa+1)} \right) + H_2[\omega'] f(\omega', \omega)^2. \quad (3.3)$$

For $\kappa = 1$ we have that

$$f(\omega', \omega)^2 = \frac{\beta[\omega] \omega'}{\beta[\omega'] \omega}, \quad (3.4)$$

where $\beta[\omega] = \sqrt{1 - \omega^2}$. We then find that the first derivative of H_p with respect to ω' evaluated at $\omega' = \omega$ is indeed zero. The second derivative evaluated at $\omega' = \omega$ leads to the following expression:

$$\frac{\partial^2 H_p}{\partial \omega'^2} \Big|_{\omega=\omega'} = - \frac{2 \left(\sqrt{1 - \omega^2} (\omega^2 - 3) + 4 \tanh^{-1} \left(\sqrt{\frac{1-\omega}{\omega+1}} \right) \right)}{g^2 \omega^2 (\omega^2 - 1)^2}. \quad (3.5)$$

This function is zero at $\omega = 0.697586$ and the second derivative is negative below this value of ω showing an instability. In our numerical simulations of the unforced NLDE [17], we find that below this value the solitary waves are metastable, with the time for the instability to set in depending on the value of ω .

IV. NLDE IN EXTERNAL ELECTROMAGNETIC FIELDS

We add electromagnetic interactions through the gauge covariant derivative

$$i\partial_\mu \Psi \rightarrow (i\partial_\mu - eA_\mu)\Psi, \quad (4.1)$$

then under the combined transformations

$$\Psi \rightarrow e^{i\Lambda(x)}\Psi; \quad A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu \Lambda, \quad \bar{\Psi} \rightarrow \bar{\Psi}e^{-i\Lambda(x)} \quad (4.2)$$

the Lagrangian is invariant. Again the conserved current is given by Eq. (2.39). The gauge invariant Lagrangian for the external field problem is

$$L = \left(\frac{i}{2} \right) [\bar{\Psi}\gamma^\mu \partial_\mu \Psi - \partial_\mu \bar{\Psi}\gamma^\mu \Psi] - m\bar{\Psi}\Psi + \frac{g^2}{\kappa + 1} (\bar{\Psi}\Psi)^{\kappa+1} - e\bar{\Psi}\gamma^\mu A_\mu \Psi. \quad (4.3)$$

One again finds that energy-momentum is conserved, with E and P given by eqs. (2.43) and (2.44). Because of Lorentz invariance it is sufficient to calculate the energy-momentum tensor in the comoving frame $v = 0$. The energy-momentum tensor in an arbitrary frame is then given by Eq. (2.45). Using the freedom of gauge invariance, one can choose the axial gauge $A^1 = 0$, $eA_0 = V(x)$. In the axial gauge the Dirac equation becomes

$$i\gamma^\mu \partial_\mu \Psi - m\Psi + g^2(\bar{\Psi}\Psi)^\kappa \Psi - \gamma^0 V(x)\Psi = 0. \quad (4.4)$$

Going into the rest frame and choosing for ψ_0 the representation of Eq. (2.5), we find that the Dirac equation becomes

$$\begin{aligned} \partial_x A + (m + \omega)B - g^2[A^2 - B^2]^\kappa B - V(x)B &= 0, \\ \partial_x B + (m - \omega)A - g^2[A^2 - B^2]^\kappa A + V(x)A &= 0. \end{aligned} \quad (4.5)$$

As in the case of the standard NLDE equation, we can again look for static solutions of the form ψ_0 , where $A = R \cos \theta$ and $B = R \sin \theta$. The conservation of T^{11} for static solutions that vanish at infinity yields the equations

$$T^{11} = \omega\psi^\dagger\psi - m\bar{\psi}\psi + L_I = 0, \quad (4.6)$$

where

$$L_I = \frac{g^2}{\kappa + 1} (\bar{\psi}\psi)^{\kappa+1} - V(x)\psi^\dagger\psi. \quad (4.7)$$

Multiplying the Dirac equation [Eq. (4.4)] on the left by $\bar{\Psi}$ and using Eq. (2.3) we obtain

$$\omega\psi^\dagger\psi + i\bar{\psi}\gamma^1 \partial_x \psi - m\bar{\psi}\psi + g^2(\bar{\psi}\psi)^{\kappa+1} - V(x)\psi^\dagger\psi = 0. \quad (4.8)$$

The energy density is given by

$$\begin{aligned} h &= \frac{i}{2} [\bar{\Psi}\gamma^1 \partial_x \Psi - \partial_x \bar{\Psi}\gamma^1 \Psi] + m\bar{\Psi}\Psi - \frac{g^2}{\kappa + 1} (\bar{\Psi}\Psi)^{\kappa+1} + V(x)\Psi^\dagger\Psi \\ &\equiv h_1 + h_2 - h_3 + h_4. \end{aligned} \quad (4.9)$$

From Eq. (4.6), (4.7) and Eq. (4.8) we again find that

$$h_3 = \frac{1}{\kappa} h_1 \quad (4.10)$$

Multiplying the Dirac equation [Eq. (4.4)] on the left by $\bar{\Psi}$ and using Eq.(2.3), eliminating the self-interaction term using Eq. (??), we obtain

$$i\bar{\psi}\gamma^1\partial_x\psi + \kappa [m\bar{\psi}\psi + V(x)\psi^\dagger\psi - \omega\psi^\dagger\psi] = 0. \quad (4.11)$$

Next using the ansatz Eq. (2.5):

$$\psi_0 = \begin{pmatrix} R(x) \cos \theta \\ i R(x) \sin \theta \end{pmatrix}, \quad (4.12)$$

we obtain the first order differential equation for θ , namely

$$\frac{d\theta}{dx} = -\kappa\omega + \kappa m \cos(2\theta) + \kappa V(x), \quad (4.13)$$

and for $R(x)$ we obtain from Eq. (4.6)

$$[\omega - V(x)]R^2 - mR^2 \cos 2\theta + \frac{g^2}{\kappa + 1} (R^2 \cos 2\theta)^{\kappa+1} = 0, \quad (4.14)$$

or

$$R^2 = \left[\frac{(\kappa + 1)(m \cos 2\theta - \omega + V(x))}{g^2 (\cos 2\theta)^{\kappa+1}} \right]^{1/\kappa}. \quad (4.15)$$

Notice, if we now choose $V(x) = \mu \cos 2\theta$ we arrive at the solutions found earlier with $m \rightarrow m + \mu$. Note that for a bound state without the external potential we needed $\omega < m$. So now we need $\omega < m + \mu$ for β_k to be real. The potential is then

$$V(x) = \mu \frac{(m + \mu) + \omega \cosh 2\beta_k x}{\omega + (m + \mu) \cosh 2\beta_k x}, \quad (4.16)$$

where

$$\beta_k = \kappa \sqrt{(m + \mu)^2 - \omega^2}. \quad (4.17)$$

The eigenvalue ω is set by fixing the charge Q of the solitary wave. We can choose a negative μ satisfying $\omega < m + \mu$ so that the potential at small x looks much like a harmonic trap. Namely, choosing $\kappa = 1, \mu = -1/4, m = 1, \omega = 1/2$, then

$$V(x) = -\frac{2 \cosh\left(\frac{\sqrt{5}x}{2}\right) + 3}{12 \cosh\left(\frac{\sqrt{5}x}{2}\right) + 8}. \quad (4.18)$$

This is plotted in Fig. 1. The charge density for the solitary wave corresponding to this external potential is plotted in Fig. 2.

At small x , $V(x)$ has the expansion:

$$V(x) \simeq -\frac{1}{4} + \frac{x^2}{32} - \frac{13x^4}{1536} + O(x^6). \quad (4.19)$$

Another solution is found if we let

$$V(x) = \mu \sin(2\theta(x)). \quad (4.20)$$

Then the solution to the differential equation:

$$\theta'(x) = -\kappa\omega + \kappa m \cos(2\theta(x)) + \kappa\mu \sin(2\theta(x)) \quad (4.21)$$

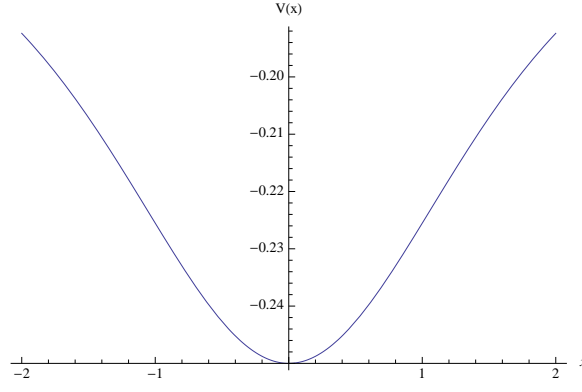


FIG. 1: (Color online) Potential $V(x)$ corresponding to Eq. (4.16) with $\mu = -1/4$, $\omega = 1/2$.

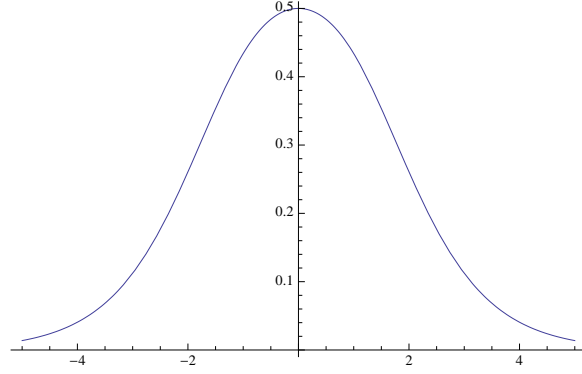


FIG. 2: (Color online) Charge density $R^2(x)$ corresponding to the potential $V(x)$ given by Eq. (4.16) with $\mu = -1/4$, $\omega = 1/2$.

is found to be

$$\theta(x) = \tan^{-1} \left(\frac{\mu + \sqrt{\mu^2 + m^2 - \omega^2} \tanh \left(\kappa (x - x_0) \sqrt{\mu^2 + m^2 - \omega^2} \right)}{m + \omega} \right). \quad (4.22)$$

Letting $x_0 = 0$, we have that

$$\tan \theta = \frac{\mu + \sqrt{\mu^2 + m^2 - \omega^2} \tanh \left(\kappa x \sqrt{\mu^2 + m^2 - \omega^2} \right)}{m + \omega}. \quad (4.23)$$

The potential is thus given by

$$\begin{aligned} V_2(x) &= \mu \sin 2\theta = 2\mu \frac{\tan \theta}{1 + \tan^2 \theta} \\ &= \frac{2\mu \left(\mu + \sqrt{\mu^2 + m^2 - \omega^2} \tanh \left(\kappa x \sqrt{\mu^2 + m^2 - \omega^2} \right) \right)}{(m + \omega) \left(\frac{\left(\mu + \sqrt{\mu^2 + m^2 - \omega^2} \tanh \left(\kappa x \sqrt{\mu^2 + m^2 - \omega^2} \right) \right)^2}{(m + \omega)^2} + 1 \right)}. \end{aligned} \quad (4.24)$$

For the values $\kappa = 1$, $\mu = -1/4$, $m = 1$, $\omega = 1/2$, we find that $V_2(x)$ has a kink-like form shown in Fig. 3. The charge density R^2 corresponding to these values of the parameters in $V_2(x)$ is shown in Fig. 4

For small x for these parameters

$$V_2(x) \simeq \frac{3}{37} - \frac{1365x}{5476} - \frac{54249x^2}{810448} + O(x^3), \quad (4.25)$$

if instead we choose $\mu = 1/4$ and leave the other parameters the same we get the opposite type of kink shown in Fig. 5. Now for small x we have

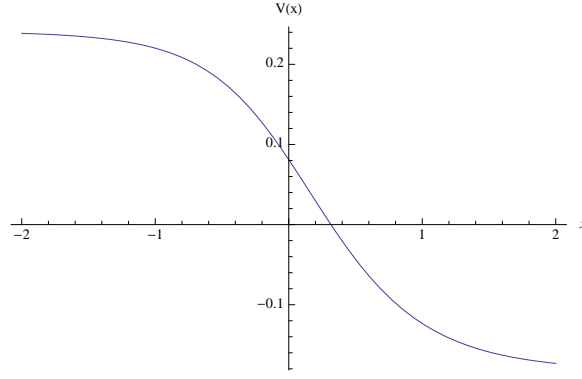


FIG. 3: (Color online) Potential $V_2(x)$ corresponding to the exact solution Eq. (4.24) with $\omega = 1/2$, $\mu = -1/4$.

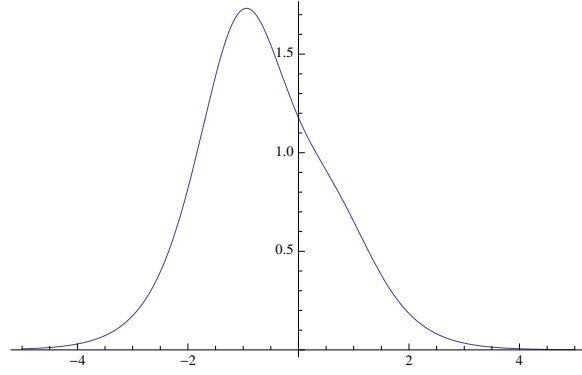


FIG. 4: (Color online) The charge density for the solitary wave in the potential $V_2(x)$ given Eq. (4.24) with $\omega = 1/2$, $\mu = -1/4$.

$$V_2(x) \simeq \frac{3}{37} + \frac{1365x}{5476} - \frac{54249x^2}{810448} + O(x^3). \quad (4.26)$$

We can also solve for the more general case

$$V(x) = \mu_1 \cos(2\theta(x)) + \mu_2 \sin(2\theta(x)), \quad (4.27)$$

then the solution to the differential equation is

$$\theta'(x) = -\kappa\omega + \kappa(m + \mu_1) \cos(2\theta(x)) + \kappa\mu_2 \sin(2\theta(x)), \quad (4.28)$$

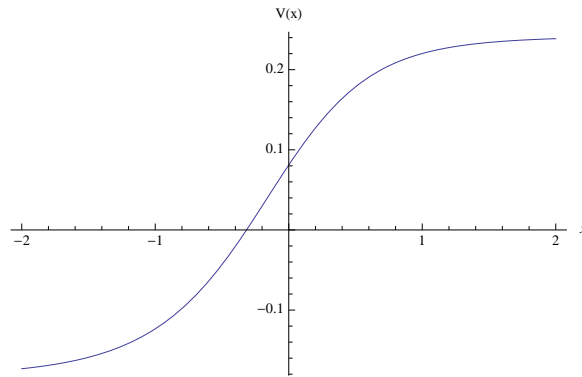


FIG. 5: (Color online) Potential $V(x)$ corresponding to the exact solution Eq. (4.24) with $\mu = 1/4$.

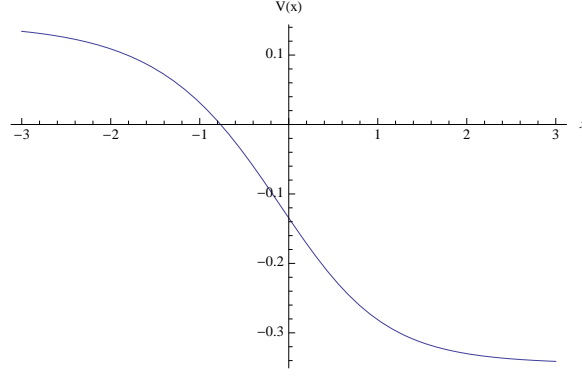


FIG. 6: (Color online) Potential $V_3(x)$ corresponding to Eq. (4.30) with $\omega = 1/2, \mu_1 = \mu_2 = -1/4$.

so that we just need to change $m \rightarrow m + \mu_1$, and $\mu \rightarrow \mu_2$ to obtain

$$\tan \theta = \frac{\mu_2 + \sqrt{\mu_2^2 + (m + \mu_1)^2 - \omega^2} \tanh \left(\kappa x \sqrt{\mu_2^2 + (m + \mu_1)^2 - \omega^2} \right)}{m + \mu_1 + \omega}. \quad (4.29)$$

The potential is then given by

$$\begin{aligned} V_3(x) &= \mu_1 \cos 2\theta + \mu_2 \sin 2\theta = \mu_1 \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} + 2\mu_2 \frac{\tan \theta}{1 + \tan^2 \theta} \\ &= \frac{N(x)}{D(x)}, \end{aligned} \quad (4.30)$$

where

$$\begin{aligned} N(x) &= \frac{2\mu_2 \left(\mu_2 + \sqrt{\mu_2^2 + (\mu_1 + m)^2 - \omega^2} \tanh \left(\kappa x \sqrt{\mu_2^2 + (\mu_1 + m)^2 - \omega^2} \right) \right)}{\mu_1 + m + \omega} \\ &+ \mu_1 \left(1 - \frac{\left(\mu_2 + \sqrt{\mu_2^2 + (\mu_1 + m)^2 - \omega^2} \tanh \left(\kappa x \sqrt{\mu_2^2 + (\mu_1 + m)^2 - \omega^2} \right) \right)^2}{(\mu_1 + m + \omega)^2} \right) \end{aligned} \quad (4.31)$$

and

$$D(x) = \frac{\left(\mu_2 + \sqrt{\mu_2^2 + (\mu_1 + m)^2 - \omega^2} \tanh \left(\kappa x \sqrt{\mu_2^2 + (\mu_1 + m)^2 - \omega^2} \right) \right)^2}{(\mu_1 + m + \omega)^2} + 1. \quad (4.32)$$

For $\omega = 1/2, \mu_1 = \mu_2 = -1/4, \kappa = 1, m = 1$, $V_3(x)$ has the form of Fig. 6. The charge density corresponding to this potential is given in Fig. 7.

One can also consider the potential $V_4 = \mu \tan^2 \theta$. In that case on substituting $t = \tan \theta$ we obtain the differential equation for t

$$\frac{dt}{dx} = \kappa (t^2(\mu - m - \omega) + m + \mu t^4 - \omega). \quad (4.33)$$

Thus we obtain

$$\begin{aligned} \kappa x &= \frac{\sqrt{2}\sqrt{\mu} \tan^{-1} \left(\frac{\sqrt{2}\sqrt{\mu} t}{\sqrt{\mu - \sqrt{(\mu + \omega)^2 + m^2 + 2m(\omega - 3\mu)} - m - \omega}} \right)}{\sqrt{(\mu + \omega)^2 + m^2 + 2m(\omega - 3\mu)} \sqrt{\mu - \sqrt{(\mu + \omega)^2 + m^2 + 2m(\omega - 3\mu)} - m - \omega}} \\ &- \frac{\sqrt{2}\sqrt{\mu} \tan^{-1} \left(\frac{\sqrt{2}\sqrt{\mu} t}{\sqrt{\mu + \sqrt{(\mu + \omega)^2 + m^2 + 2m(\omega - 3\mu)} - m - \omega}} \right)}{\sqrt{(\mu + \omega)^2 + m^2 + 2m(\omega - 3\mu)} \sqrt{\mu + \sqrt{(\mu + \omega)^2 + m^2 + 2m(\omega - 3\mu)} - m - \omega}}. \end{aligned} \quad (4.34)$$

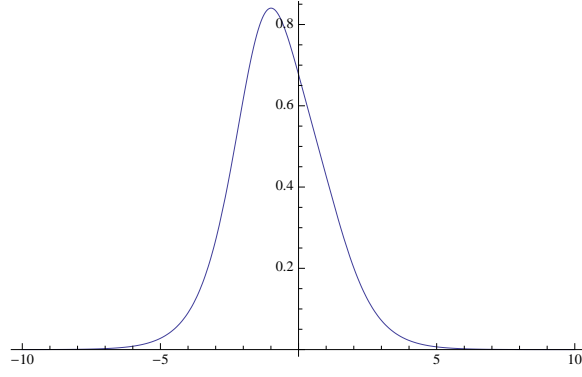


FIG. 7: (Color online) Charge density for the solitary wave in the potential $V_3(x)$ corresponding to Eq. (4.30) with $\omega = 1/2, \mu_1 = \mu_2 = -1/4$.

A. Stability of exact solutions with $\kappa = 1$ in the presence of an external field

Again we follow the method of Bogolubsky [16] and consider stability to changes in frequency keeping the charge Q fixed. That is if we parametrize a rest frame solitary wave solution by

$$\psi_s(x, t) = \chi_s(x, \omega) e^{-i\omega t}, \quad (4.35)$$

then we choose our perturbed wave function to be

$$\begin{aligned} \tilde{\psi}[x, t, \omega', \omega] &= \frac{\sqrt{Q[\omega]}}{\sqrt{Q[\omega']}} \chi_s(x, \omega') e^{-i\omega' t} \\ &\equiv f(\omega', \omega) \chi_s(x, \omega') e^{-i\omega' t}. \end{aligned} \quad (4.36)$$

Again inserting this wave function into the Hamiltonian we get a new probe Hamiltonian H_p depending on both ω', ω . We will consider the case where the potential has the form

$$V(x) = \mu_1 \cos 2\theta + \mu_2 \sin 2\theta. \quad (4.37)$$

The energy density is given by the four terms in Eq. (4.9), $(h_1 + h_2 - h_3 + h_4)$, where

$$\begin{aligned} h_1 &= R^2 \frac{d\theta}{dx}; \quad h_2 = mR^2 \cos 2\theta, \\ h_3 &= \frac{g}{2} (R^2 \cos 2\theta)^2; \quad h_4 = R^2 (\mu_1 \cos 2\theta + \mu_2 \sin 2\theta). \end{aligned}$$

We also have that $h_3 = h_1$. Following our previous discussion, the probe Hamiltonian has the form:

$$H_p[\omega', \omega] = H_1[\omega'] (f(\omega', \omega)^2 - f(\omega', \omega)^4) + (H_2 + H_4)[\omega'] f(\omega', \omega)^2. \quad (4.38)$$

For our choice of potential, we have the following relationship:

$$\frac{d\theta}{dx} = (m + \mu_1) \cos 2\theta + \mu_2 \sin 2\theta - \omega. \quad (4.39)$$

Solving this equation we obtain

$$T \equiv \tan \theta = D + E \tanh Bx, \quad (4.40)$$

where

$$B = \sqrt{(m + \mu_1)^2 + \mu_2^2 - \omega^2}; \quad E = \frac{B}{d_1}; \quad D = \frac{\mu_2}{d_1}; \quad d_1 = m + \mu_1 + \omega. \quad (4.41)$$

We also have

$$\sin 2\theta = \frac{2T}{1 + T^2}; \quad \cos 2\theta = \frac{1 - T^2}{1 + T^2}, \quad (4.42)$$

therefore

$$\begin{aligned} R^2 &= \frac{2}{g^2} \left[\frac{(m + \mu_1) \cos 2\theta + \mu_2 \sin 2\theta - \omega}{\cos^2 2\theta} \right] \\ &= \frac{2B^2}{d_1 g^2} \operatorname{sech}^2 Bx \frac{[1 + (D + E \tanh Bx)^2]}{[1 - (D + E \tanh Bx)^2]^2}. \end{aligned} \quad (4.43)$$

The charge is given by

$$Q[\omega] = \int_{-\infty}^{\infty} dx R^2 = \frac{2\omega \sqrt{(m + \mu_1)^2 + \mu_2^2 - \omega^2}}{g^2(\omega^2 - \mu_2^2)}, \quad (4.44)$$

thus

$$f(\omega', \omega)^2 = \frac{Q[\omega]}{Q[\omega']}. \quad (4.45)$$

Now since

$$h_1 = R^2[(m + \mu_1) \cos 2\theta + \mu_2 \sin 2\theta - \omega] \quad (4.46)$$

and

$$h_2 + h_4 = R^2[(m + \mu_1) \cos 2\theta + \mu_2 \sin 2\theta], \quad (4.47)$$

we find

$$H_1 = -\omega Q + H_2 + H_4 \quad (4.48)$$

and

$$H_2 + H_4 = (m + \mu_1)T_1 + \mu_2 T_2, \quad (4.49)$$

where

$$\begin{aligned} T_1 &= \int dx R^2 \cos 2\theta = 2(\tanh^{-1}(E - D) + \tanh^{-1}(E + D)) \\ &= \frac{2}{g^2} \tanh^{-1} \left(\frac{\sqrt{\mu_1^2 + \mu_2^2 + m^2 + 2\mu_1 m - \omega^2}}{\mu_1 + m} \right), \end{aligned} \quad (4.50)$$

$$T_2 = \int dx R^2 \sin 2\theta = \frac{2\mu_2(m + \mu_1)}{g^2(\omega^2 - \mu_2^2)} - \frac{2\mu_2}{g^2} \frac{\tanh^{-1} \left(\frac{\sqrt{m^2 + 2m\mu_1 + \mu_1^2 + \mu_2^2 - \omega^2}}{m + \mu_1} \right)}{\sqrt{(m + \mu_1)^2 + (\mu_2^2 - \omega^2)}}. \quad (4.51)$$

Thus

$$H_p[\omega'] = -\omega' Q[\omega'] f(\omega', \omega)^2 (1 - f(\omega', \omega)^2) + f(\omega', \omega)^2 (H_2[\omega'] + H_4[\omega']) (2 - f(\omega', \omega)^2). \quad (4.52)$$

If we take the first derivative of this probe Hamiltonian with respect to ω' and set $\omega' = \omega$ we find that this is zero only when $\mu_2 = 0$. Thus the probe Hamiltonian is stationary to ω variations only when $\mu_2 = 0$. However, we can use this method for the external potential $V_1(x)$. In that case the first derivative is automatically zero when $\omega' = \omega$. The second derivative changes sign at a particular value of $\omega = \omega^*$, which for $m = 1$ is the solution to the equation:

$$\frac{2(\mu_1 + 1)^3 \coth^{-1} \left(\frac{\mu_1 + 1}{\sqrt{\mu_1(\mu_1 + 2) - \omega^2 + 1}} \right)}{\sqrt{\mu_1^2 + 2\mu_1 - \omega^2 + 1}} - 3\mu_1^2 - 6\mu_1 + \omega^2 - 3 = 0. \quad (4.53)$$

Now we have that for there to be an allowed real solution $\omega < m + \mu$. So if μ is positive, the original ω_c is shifted upward and so is the region in ω space where the solutions are real. If μ is negative ω_c is shifted downward, but also the allowed regime of ω is decreased. The net result is that the possible stability of stability is approximately 30% of the allowed region for real solutions independent of the value of μ . However, preliminary simulations show that some of the exact solutions with the potential (4.16) are metastable.

V. VARIATIONAL ANSATZ FOR THE NLDE IN EXTERNAL FIELDS

The gauge invariant Lagrangian for the external field problem is

$$L = \left(\frac{i}{2}\right) [\bar{\Psi}\gamma^\mu\partial_\mu\Psi - \partial_\mu\bar{\Psi}\gamma^\mu\Psi] - m\bar{\Psi}\Psi + \frac{g^2}{\kappa+1}(\bar{\Psi}\Psi)^{\kappa+1} - e\bar{\Psi}\gamma^\mu A_\mu\psi. \quad (5.1)$$

Using the freedom of gauge invariance, one can choose the axial gauge $A^1 = 0$, $eA_0 = V(x)$.

Our ansatz for the trial variational wave function is to assume that because of the smallness of the perturbation the main modification to our exact solutions to the NLDE equation without an external field is that the parameters describing the position, momentum, boost and phase become time dependent.

That is, we replace

$$vt \rightarrow q(t); \quad \eta \rightarrow \eta(t); \quad \gamma\omega v \rightarrow p(t); \quad \omega t' = \gamma\omega(t - vx) \rightarrow \phi(t) - p(t)(x - q(t)), \quad (5.2)$$

where $\phi(t) = \omega\gamma t - p(t)q(t)$.

Thus our trial wave function in component form is given by:

$$\begin{aligned} \Psi_1(x, t) &= \left(\cosh \frac{\eta}{2} A(x') + i \sinh \frac{\eta}{2} B(x')\right) e^{-i\phi + ip(x-q)}, \\ \Psi_2(x, t) &= \left(\sinh \frac{\eta}{2} A(x') + i \cosh \frac{\eta}{2} B(x')\right) e^{-i\phi + ip(x-q)}, \end{aligned} \quad (5.3)$$

where $x' = \cosh \eta(t) (x - q(t))$. Using this trial wave function we can determine the effective Lagrangian for the variational parameters. Writing the Lagrangian density as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3, \quad (5.4)$$

where

$$\begin{aligned} \mathcal{L}_1 &= \frac{i}{2} (\bar{\Psi}\gamma^\mu\partial_\mu\Psi - \partial_\mu\bar{\Psi}\gamma^\mu\Psi) \\ &= \text{Re} (i\Psi^\dagger\partial_t\Psi + i\Psi^\dagger\sigma_1\partial_x\Psi) \\ &= (p\dot{q} + \dot{\phi} - \dot{p}(x-q)) (A^2(x') + B^2(x')) \cosh \eta + \dot{q} \sinh \eta \cosh \eta (A(x')B'(x') - B(x')A'(x')) \\ &\quad + (x-q)\dot{\eta} \sinh^2 \eta (BA' - AB') - p \sinh \eta (A^2 + B^2) - \cosh^2 \eta (AB' - BA'). \end{aligned} \quad (5.5)$$

Here $B'(x') = \frac{dB(x')}{dx'}$. Integrating over x and changing integration variables to $z = (x - q) \cosh \eta$ one obtains

$$L_1 = \int dx \mathcal{L}_1 = Q (p\dot{q} + \dot{\phi} - p \tanh \eta) - I_0 (\cosh \eta - \dot{q} \sinh \eta), \quad (5.6)$$

where

$$Q = \int dz [A^2(z) + B^2(z)], \quad (5.7)$$

is as given by Eq. (2.41). The second integral is

$$I_0 = \int dz (B'A - A'B) = H_1, \quad (5.8)$$

where H_1 is the rest frame kinetic energy and is given by Eq. (2.51).

Now let us turn to L_2 . We have

$$\mathcal{L}_2 = -m\bar{\Psi}\Psi + \frac{g^2}{\kappa+1}(\bar{\Psi}\Psi)^{\kappa+1}; \quad \bar{\Psi}\Psi = A^2(x') - B^2(x'), \quad (5.9)$$

thus

$$L_2 = \int \mathcal{L}_2 dx = -\frac{m}{\cosh \eta} I_1 + \frac{g^2}{(\kappa+1) \cosh \eta} I_2, \quad (5.10)$$

where

$$I_1 = \int dz (A^2(z) - B^2(z)), \quad (5.11)$$

$$I_2 = \int dz (A^2(z) - B^2(z))^{(\kappa+1)}. \quad (5.12)$$

Finally

$$\mathcal{L}_3 = -eA_0 \bar{\Psi} \gamma^0 \Psi \equiv -V(x) \Psi^\dagger \Psi = -V(x) \cosh \eta [A^2(x') + B^2(x')] = -V(x) \cosh \eta \rho(x'), \quad (5.13)$$

thus

$$L_3 = - \int dz \rho(z) V \left[\frac{z}{\cosh \eta} + q(t) \right] = -U[\eta(t), q(t)]. \quad (5.14)$$

Putting these terms together we obtain:

$$\begin{aligned} L = & Q(p\dot{q} + \dot{\phi} - p \tanh \eta) - I_0 (\cosh \eta - \dot{q} \sinh \eta) \\ & - \frac{m}{\cosh \eta} I_1 + \frac{g^2}{(\kappa+1) \cosh \eta} I_2 - U[\eta(t), q(t)]. \end{aligned} \quad (5.15)$$

We now get the following Lagrange's equations:

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{\phi}} = 0 \rightarrow \frac{dQ}{dt} = 0, \rightarrow Q = \text{const}, \quad (5.16)$$

i.e. the charge is canonically conjugated to the phase ϕ . The canonical soliton momentum, which is conjugated to the soliton position, is

$$\begin{aligned} P_q &= \frac{\delta L}{\delta \dot{q}} = Qp + I_0 \sinh \eta; \\ \frac{dP_q}{dt} &= Q\dot{p} + I_0 \cosh \eta \dot{\eta} = \frac{\delta L}{\delta q} = -\frac{\partial U}{\partial q}. \end{aligned}$$

From

$$\frac{\delta L}{\delta p} = 0 \rightarrow \dot{q} = \tanh \eta; \quad (5.17)$$

which implies $\sinh \eta = \gamma \dot{q}$ and $\cosh \eta = \gamma = (1 - \dot{q}^2)^{-1/2}$.

$$\begin{aligned} \frac{\delta L}{\delta \eta} = 0 \rightarrow & -Qp \operatorname{sech}^2 \eta - I_0 (\sinh \eta - \dot{q} \cosh \eta) \\ & + \tanh \eta \operatorname{sech} \eta \left(mI_1 - \frac{g^2}{(\kappa+1)} I_2 \right) - \frac{\partial U}{\partial \eta} = 0. \end{aligned} \quad (5.18)$$

Changing variables to $\dot{q} = \tanh \eta$ and using

$$\frac{d\dot{q}}{d\eta} = \operatorname{sech}^2 \eta, \quad (5.19)$$

we obtain

$$Qp(t) = \gamma \dot{q} \left(mI_1 - \frac{g^2}{(\kappa+1)} I_2 \right) - \frac{\partial U}{\partial \dot{q}} \quad (5.20)$$

and

$$Q\dot{p} + I_0 \gamma \dot{q} = \frac{\delta L}{\delta q} = -\frac{\partial U}{\partial q}. \quad (5.21)$$

From Eq. (5.20) we also have

$$Q\dot{p} = \frac{d(\gamma\dot{q})}{dt} \left(mI_1 - \frac{g^2}{(\kappa+1)} I_2 \right) - \frac{d}{dt} \frac{\partial U}{\partial \dot{q}}. \quad (5.22)$$

Combining Eqs. (5.22), (5.21) we obtain:

$$\mu \frac{d(\gamma\dot{q})}{dt} = F_{eff}[q, \dot{q}], \quad (5.23)$$

where

$$\mu = mI_1 - \frac{g^2}{(\kappa+1)} I_2 + I_0; \quad F_{eff}[q, \dot{q}] = \frac{d}{dt} \frac{\partial U}{\partial \dot{q}} - \frac{\partial U}{\partial q}. \quad (5.24)$$

Now for the NLDE without the presence of external forces, the solitary wave in the frame with $v = 0$ obeys the relationship [4]:

$$\omega\psi^\dagger\psi - m\bar{\psi}\psi + \frac{g^2}{\kappa+1}(\bar{\psi}\psi)^{\kappa+1} = 0. \quad (5.25)$$

For our problem this converts into

$$\omega(A^2 + B^2) - m(A^2 - B^2) + \frac{g^2}{(\kappa+1)}(A^2 - B^2)^{(\kappa+1)} = 0. \quad (5.26)$$

Integrating this relationship we obtain:

$$mI_1 - \frac{g^2}{(\kappa+1)} I_2 = \omega Q, \quad (5.27)$$

thus we can write Eq. (5.23) as

$$\frac{d(M\dot{q})}{dt} = F_{eff}[q, \dot{q}], \quad (5.28)$$

where

$$M = (Q\omega + I_0)\gamma = M_0\gamma. \quad (5.29)$$

Here F_{eff} is given by Eq. (5.24), where

$$U(\dot{q}, q) = \int_{-\infty}^{\infty} dz V \left(q + \frac{z}{\gamma} \right) [A^2(z) + B^2(z)], \quad (5.30)$$

and $\gamma = 1/\sqrt{1 - \dot{q}^2}$. We can rewrite the relativistic force equation as

$$\gamma^3 M_0 \ddot{q} = F_{eff}[q, \dot{q}]. \quad (5.31)$$

Using the rest frame identities of the Appendix we have that

$$M_0 = I_0 + \omega Q. \quad (5.32)$$

Here I_0 is given by Eq. (5.8), and Q is given by Eq. (2.41).

It is useful to rewrite the equation for the canonical momentum P_q using the definition of M_0 and Eq. (5.20) as follows:

$$P_q = Qp + I_0\gamma\dot{q} = M_0\gamma\dot{q} - \frac{\partial U[q, \dot{q}]}{\partial \dot{q}}. \quad (5.33)$$

A. Energy-momentum tensor

The fact that the external potential is explicitly independent of time means that the energy of the solitary wave is independent of time. The energy density is given by

$$T^{00} = \frac{i}{2}(\bar{\psi}\gamma^0\partial_t\psi - \partial_t\bar{\psi}\gamma^0\psi) - \mathcal{L}. \quad (5.34)$$

Straightforward integration leads to

$$E = \int dx T^{00} = Qp\dot{q} + \gamma I_0 + \frac{m}{\gamma}I_1 - \frac{g^2}{(\kappa+1)\gamma}I_2 + U[q, \dot{q}]. \quad (5.35)$$

Using the identities Eq. (5.27) and Eq.(5.29) we can rewrite this as

$$E = M_0\gamma + Qp\dot{q} - \gamma\omega Q\dot{q}^2 + U[q, \dot{q}]. \quad (5.36)$$

From (5.20) and (5.27) we have that

$$Qp = \gamma\dot{q}\omega Q - \frac{\partial U}{\partial \dot{q}}, \quad (5.37)$$

thus we can write the energy of the solitary wave in the convenient form:

$$E = M_0\gamma + U[q, \dot{q}] - \dot{q}\frac{\partial U}{\partial \dot{q}}. \quad (5.38)$$

The conservation of energy will be important to test our numerical integration schemes in Sec. VII.

For time independent external forces the total momentum of the solitary wave is not conserved but changes depending on the external force. We have that

$$P = \int T^{01}dx = -\frac{i}{2}\int dx(\psi^\dagger\partial_x\psi - \partial_x\psi^\dagger\psi). \quad (5.39)$$

Explicitly we obtain

$$P = \gamma\dot{q}I_0 + pQ, \quad (5.40)$$

where I_0 is given by Eq. (5.8). Using Eq. (5.20), we can rewrite this as

$$P = \gamma M_0\dot{q} - \frac{\partial U[q, \dot{q}]}{\partial \dot{q}}, \quad (5.41)$$

which we recognize as identical to the canonical momentum $P_q = \frac{\delta L}{\delta \dot{q}}$ given by Eq. (5.33). The Lagrange equation for P_q is

$$\dot{P}_q = -\frac{\partial U[q, \dot{q}]}{\partial q} = \frac{d}{dt}\left(M_0\gamma\dot{q} - \frac{\partial U[q, \dot{q}]}{\partial \dot{q}}\right). \quad (5.42)$$

VI. STABILITY CONJECTURE

For the NLSE it was shown earlier [9, 11] that a reliable dynamical stability criterion for the breakup of the solitary wave under external forces was that the solitary wave will be stable if

$$\frac{\partial p(t)}{\partial \dot{q}(t)} > 0. \quad (6.1)$$

Here $p(t)$ is the normalized momentum of the solitary wave $P(t)/M(t)$, where $M = \int dx \Psi^*(x, t)\Psi(x, t)$ is the “mass” of the solitary wave. For the NLDE Q takes the place of M . However, Q is a conserved variable so one can use the canonical momentum $P(t)$ instead of $P(q, \dot{q})/Q$ to study stability. Using our collective coordinate theory, this leads

to the criterion that a *necessary* (but not sufficient) condition for stability of the solitary waves of the CC theory is that

$$\frac{\partial P(q, \dot{q})}{\partial \dot{q}} = \gamma^3 M_0 - \frac{\partial^2 U}{\partial \dot{q}^2} > 0. \quad (6.2)$$

Note that the r.h.s. of Eq. (6.2) plays the role of a time dependent mass. The *sufficient* condition for the solitary wave solution to the CC equation to be *unstable* in our simulations is that

$$\frac{\partial P(q, \dot{q})}{\partial \dot{q}} < 0. \quad (6.3)$$

Following Comech's reasoning [14], we expect that in the nonrelativistic regime where ω is close to m that this criteria will be valid in determining stability.

VII. SIMPLE EXTERNAL POTENTIALS

A. Simulations

The numerical simulations have been performed by means of a 4th order Runge-Kutta method. We choose $N + 1$ points starting at $n = 0$ and vanishing boundary conditions $\Psi(\pm L, t) = 0$. The other parameters related with the discretization of the system are $x \in [-100, 100]$, $\Delta x = 0.02$, $\Delta t = 0.0001$. For our initial conditions on the soliton, we use the exact 1-soliton solutions of the unforced nonlinear Dirac equation discussed in Sec. II. Since we would like to compare the exact numerical solution with the solution of the CC equations, we need to define how we determine the position of the soliton. In our numerical computation of $q(t)$ we have used the first moment of the charge, i.e.

$$q(t) = Q_1(t)/Q(t), \quad (7.1)$$

where

$$Q_1 = \int dx \, x \, \Psi^\dagger(x) \Psi(x), \quad Q = \int dx \, \Psi^\dagger(x) \Psi(x), \quad (7.2)$$

B. linear potential (ramp potential)

Consider the constant external force with scalar potential $V(x) = -V_1 x$, and $V_1 > 0$. We then have from Eq. (5.14) that

$$U = -V_1 q(t) Q, \quad (7.3)$$

the force law then becomes

$$\frac{d}{dt}(M_0 \gamma \dot{q}) = V_1 Q. \quad (7.4)$$

Integrating once (starting at an initial velocity \dot{q}_0) one has

$$\frac{\dot{q}}{\sqrt{1 - \dot{q}^2}} = c_1 t + c_2, \quad (7.5)$$

where $c_1 = V_1 Q/M_0$ and $c_2 = \frac{\dot{q}_0}{\sqrt{1 - \dot{q}_0^2}}$. Integrating we obtain

$$q(t) = \frac{\sqrt{(c_1 t + c_2)^2 + 1}}{c_1} - \frac{\sqrt{c_2^2 + 1}}{c_1} + q(0). \quad (7.6)$$

This is the standard result for a relativistic point particle undergoing constant acceleration. If we choose $q(0) = \dot{q}(0) = 0$, we get the simpler expression

$$q(t) = \frac{1}{c_1} \left[\sqrt{1 + c_1^2 t^2} - 1 \right]. \quad (7.7)$$

The energy of the solitary wave is just

$$E = M_0\gamma + U = M_0, \quad (7.8)$$

and the force law is now

$$\dot{P} = -\frac{\partial U}{\partial q} = V_1 Q, \quad (7.9)$$

so that

$$P = V_1 Q t. \quad (7.10)$$

Since U in this case is independent of \dot{q} , we find from Eq. (6.2) that

$$\frac{\partial P}{\partial \dot{q}} = \gamma^3 M_0 > 0, \quad (7.11)$$

thus the *necessary* condition for stability is fulfilled. In this section and what follows we will confine ourselves to the case where $\kappa = 1$ and also $Q = 1$. For that case from the Bogolubsky stability requirement [16] we know that without forcing when $\omega < \omega_c = 0.697586$, the solitary waves are unstable.

Let us look at the case where the unforced solitary wave is stable. For $\omega = 0.9, g = 1$ we have solved numerically the NLDE for $V_1 = 0.01$ (as well as $V_1 = 10^{-3}$ and $V_1 = 10^{-4}$). We find for all these values of V_1 the solitary wave is stable at all simulation times and the center of the solitary wave follows the analytic formula we derived from the CC equation for $q(t)$, namely Eq. (7.7). In Fig. 8 we display the results of the simulation for $\rho(t), q(t), P(t)$ for $V_1 = 0.01$. We notice that the width of the soliton gets Lorentz contracted as the velocity increases (this effect is not apparent for the smaller values of V_1). Because the charge is conserved, the height of the solitary wave increases due to the increase of $\gamma(t)$.

For the case $\omega = 0.3$, the unforced solitary wave has double humped behavior and is unstable at late times. Here our simulations show that until the instability sets in (around $t \approx 110$, for $m = 1, g = 1, V_1 = 0.01$) the position of the solitary wave follows the analytic solution of the CC equation Eq. (7.7). However, the actual shape of the solitary wave becomes asymmetric with the left hump becoming higher than the right hump as a precursor for the wave becoming unstable. This is shown in Fig. 9, where $\rho_Q(x, t_{fixed})$ is plotted against x for various $t = t^*$. In Fig. 10 we give results of the simulation for the case where $\omega = 0.3, V_1 = 0.0001$. Here, looking at $q(t)$ we explicitly see that around $t = 120$, the solution of the NLDE diverges from the solution of the CC equation. Also for this value of the potential the solitary wave humps are symmetric and that the single solitary wave breaks up into two solitary waves with some radiation when it goes unstable.

C. Harmonic Potential

Let us consider the case of an external harmonic potential, $V(x) = \frac{1}{2}V_2x^2, V_2 > 0$. For that case from Eq. (5.14) we find that

$$\begin{aligned} U &= \frac{1}{2}V_2 \int_{-\infty}^{\infty} dz \left[\left(q + \frac{z}{\gamma} \right)^2 [A^2(z) + B^2(z)] \right] \\ &= \frac{1}{2}V_2 [Qq^2(t) + (1 - \dot{q}^2(t)) I_3], \end{aligned} \quad (7.12)$$

where $I_3 = \int_{-\infty}^{\infty} dz \ z^2 [A^2(z) + B^2(z)]$. From Eq. (5.24) we have

$$F_{eff}[q, \dot{q}] = -V_2 I_3 \ddot{q} - V_2 Q q \quad (7.13)$$

leading to the equation of motion [see Eq. (5.28)]:

$$\frac{d}{dt} \{ [M_0\gamma[\dot{q}(t)] + V_2 I_3] \dot{q} \} = -V_2 q(t) Q. \quad (7.14)$$

This can be rewritten as

$$[M_0\gamma^3 + V_2 I_3] \ddot{q} + V_2 Q q = 0. \quad (7.15)$$

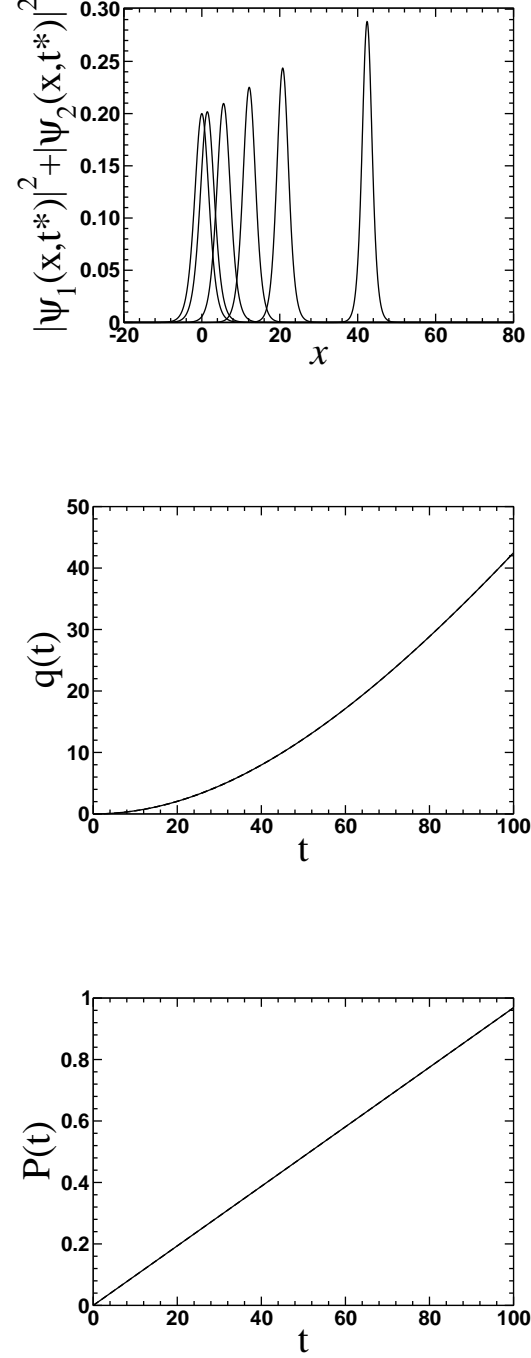


FIG. 8: Simulations with a ramp potential, $V(x) = -V_1 x$. Upper panel: Charge density at $t^* = 16.6; 33.3; 50; 66.6; 83.3; 100$. Middle and lower panels: soliton position $q(t)$ and momentum $P(t)$ from analytical results of the CC equations (solid lines) and from numerical simulations (dashed lines) of the forced NLDE. The curves are super-imposed. For the final time of integration the relative error of $q(t)$ is of order 10^{-5} . Parameters: $g = 1$, $m = 1$, $\omega = 0.9$ and $V_1 = 0.01$. Initial condition: exact soliton of the unperturbed NLDE with zero initial velocity.

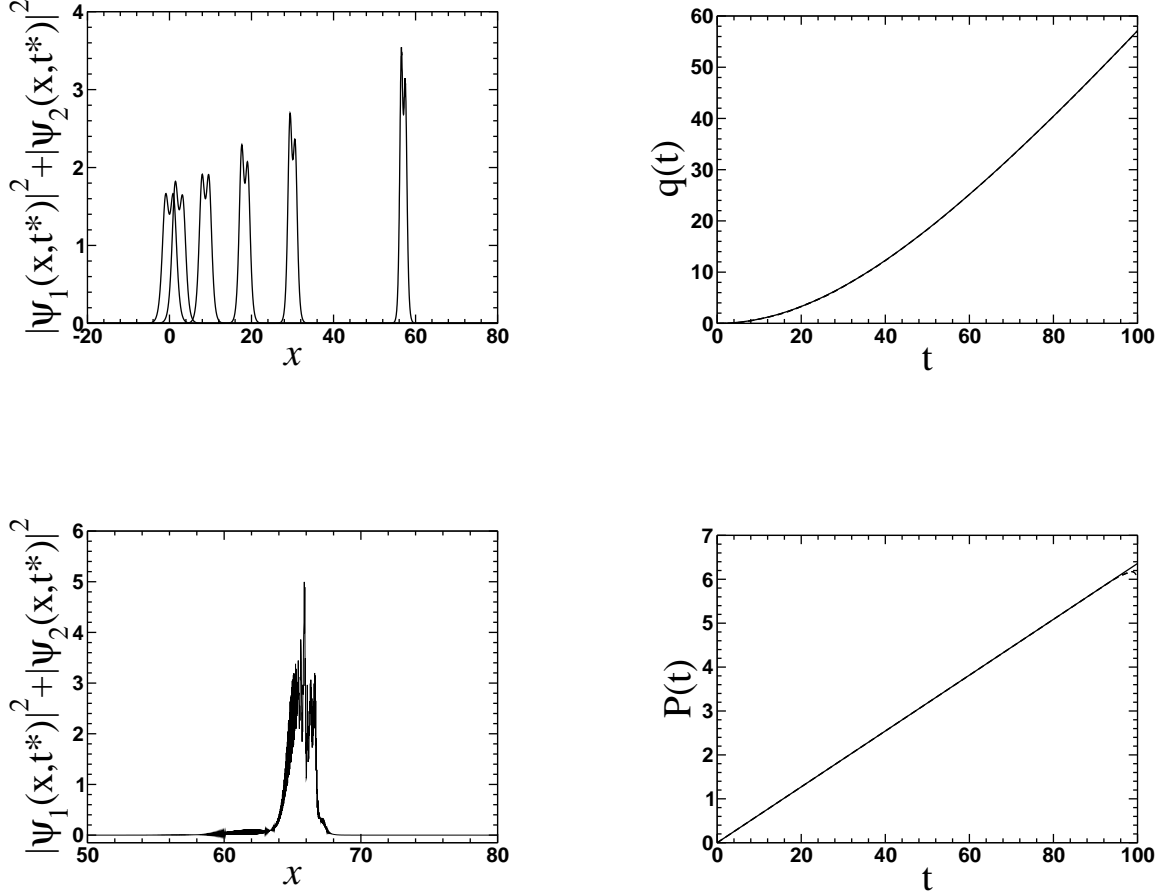


FIG. 9: Simulations with a ramp potential, $V(x) = -V_1x$ with ω in the unstable regime. Left upper panel: Charge density at $t^* = 16.6; 33.3; 50; 66.6; 83.3; 100$. Left lower panel: Charge density at $t^* = 110$. Right panels: $q(t)$ and $P(t)$ from analytical results of the CC equations (solid lines) and from numerical simulations of the forced NLDE (dashed lines), the curves are super-imposed. Parameters: $g = 1$, $m = 1$, $\omega = 0.3$ and $V_1 = 0.01$. Initial condition: exact soliton of the unperturbed NLDE with zero initial velocity.

In the nonrelativistic regime where $\gamma[\dot{q}(t)] \approx 1$ we recover the oscillator equation for the collective coordinate $q(t)$ namely

$$\ddot{q} + \Omega^2 q = 0, \quad \Omega^2 = \frac{V_2 Q}{M_0 + V_2 I_3}. \quad (7.16)$$

Note that the rest mass is *increased* by the term $V_2 I_3 > 0$. For initial conditions $q_0 = 0$, $\dot{q}(0) = v_0$ we obtain

$$q(t) = \frac{v_0}{\Omega} \sin \Omega t. \quad (7.17)$$

1. Energy conservation

From the energy conservation equation in the form of Eq. (5.38)

$$E = M_0 \gamma + U[q, \dot{q}] - \dot{q} \frac{\partial U}{\partial \dot{q}} \quad (7.18)$$

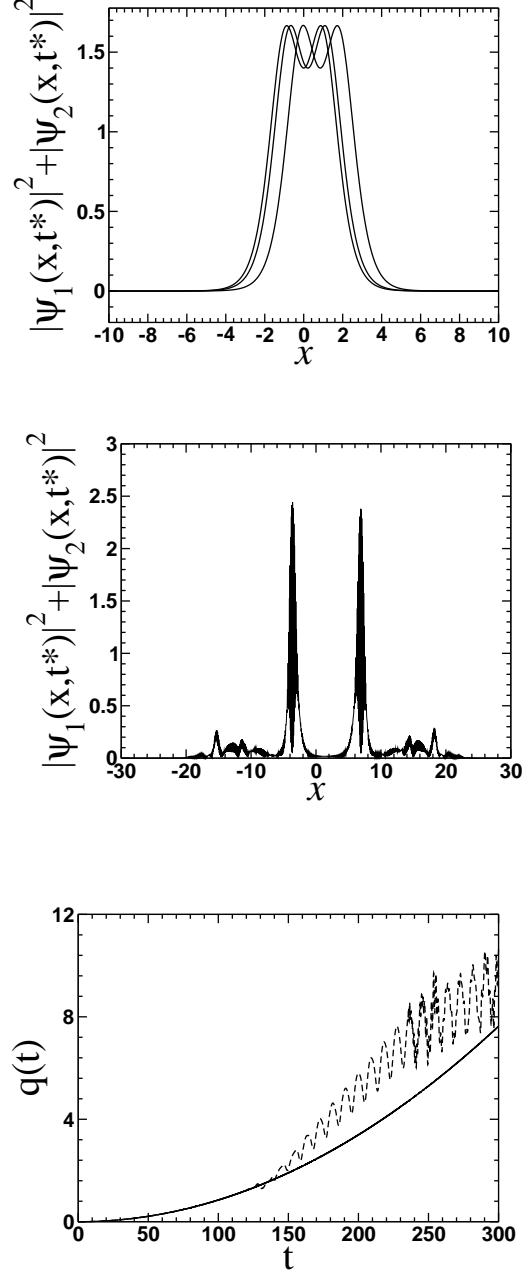


FIG. 10: Simulations with a ramp potential, $V(x) = -V_1x$ with ω in the unstable regime. Upper panel: Charge density at $t^* = 0; 50; 100$. Middle panel: Charge density at $t^* = 150$. Lower panel: $q(t)$ from analytical results of the CC equations (solid line) and from numerical simulations (dashed line) of the forced NLDE. Parameters: $g = 1$, $m = 1$, $\omega = 0.3$ and $V_1 = 0.0001$. Initial condition: exact soliton of the unperturbed NLDE with zero initial velocity.

we obtain

$$E = M_0\gamma + \frac{1}{2}QV_2q^2 + \frac{1}{2}V_2I_3(1 + \dot{q}^2). \quad (7.19)$$

In the low velocity limit, we need to keep the first two terms in the expansion of γ in the expression for the energy; namely (here we suppress the speed of light in the v/c expansion)

$$\gamma = 1 + \frac{1}{2}\dot{q}^2 + \dots \quad (7.20)$$

We then have for the solution $q(t) = \frac{v_0}{\Omega} \sin \Omega t$ the usual equipartition of energy and that the non-relativistic energy is twice the initial kinetic energy (apart from a constant)

$$E = M_0 + 2T; \quad T = (M_0 + V_2 I_3) v_0^2 \quad (7.21)$$

with the effective mass of the solitary wave increased over the unforced case by the quantity $V_2 I_3$.

2. Canonical momentum and stability Criterion

From the equation for the canonical momentum, Eq. (5.41), we find

$$P = (M + V_2 I_3) \dot{q} = (M_0 \gamma + V_2 I_3) \dot{q}. \quad (7.22)$$

This again shows the mass increased by $V_2 I_3$. The stability criterion Eq. (6.2), leads to

$$\frac{\partial P(q, \dot{q})}{\partial \dot{q}} = \gamma^3 M_0 + V_2 I_3 > 0. \quad (7.23)$$

Thus the necessary condition for stability of the solitary wave is fulfilled.

We would now like to see how well the CC equations for q and P , namely Eqs. (7.15) and (7.22), compare with the numerical solutions of the forced nonlinear Dirac equation. We will choose our initial condition to be $q(0) = 0, \dot{q}(0) = v_0$ and study both the nonrelativistic regime ($v_0 = 0.1$) and the relativistic regime ($v_0 = 0.9$). The external potential can be written as

$$V = \frac{1}{2} V_2 x^2 = \frac{1}{2} \left(\frac{x}{l} \right)^2, \quad (7.24)$$

which identifies the characteristic length of the potential as $l = 1/\sqrt{V_2}$. We would like to choose the characteristic length of the potential to be large compared to the width of the solitary wave which is $1/(2\beta)$, $\beta = \sqrt{1 - \omega^2}$. Choosing $V_2 = 10^{-4}$ accomplishes this requirement. At low velocities both $P(t)$ and $\dot{q}(t)$ are proportional to $\cos \Omega t$, thus $P(\dot{q})$ is a straight line with positive slope.

First let us consider the regime where the unforced solitary waves are stable, and choose $\omega = 0.9, g = 1$. In the non-relativistic regime ($v_0 = 0.1$) we get the results shown in Fig. 11. The oscillations of both $q(t)$ and $P(t)$ are harmonic as predicted by the CC equations. The charge density maintains its shape as its position oscillates periodically in time. For $v_0 = 0.9$, q and P again follow the CC equations for a little less than half the oscillation period but then the solitary wave becomes unstable and the exact simulation of q and P then diverges from the solution to the CC equations. This is shown in Fig. 12.

Next we consider the regime $0 < \omega < \omega_c = 0.697586$ where the unforced soliton is unstable. For the parameters $\omega = 0.3, g = 1, V_2 = 10^{-4}$, we obtain the typical result found in the unforced problem that at around $t = 120$, the solitary wave becomes unstable. Until then the CC equations for $q(t)$ and $P(t)$ track well the exact solution. This is seen in Fig. 13. However, the wave function starts becoming asymmetric at late times and departs from our symmetric ansatz even before the soliton becomes unstable and breaks into two solitary waves plus some radiation.

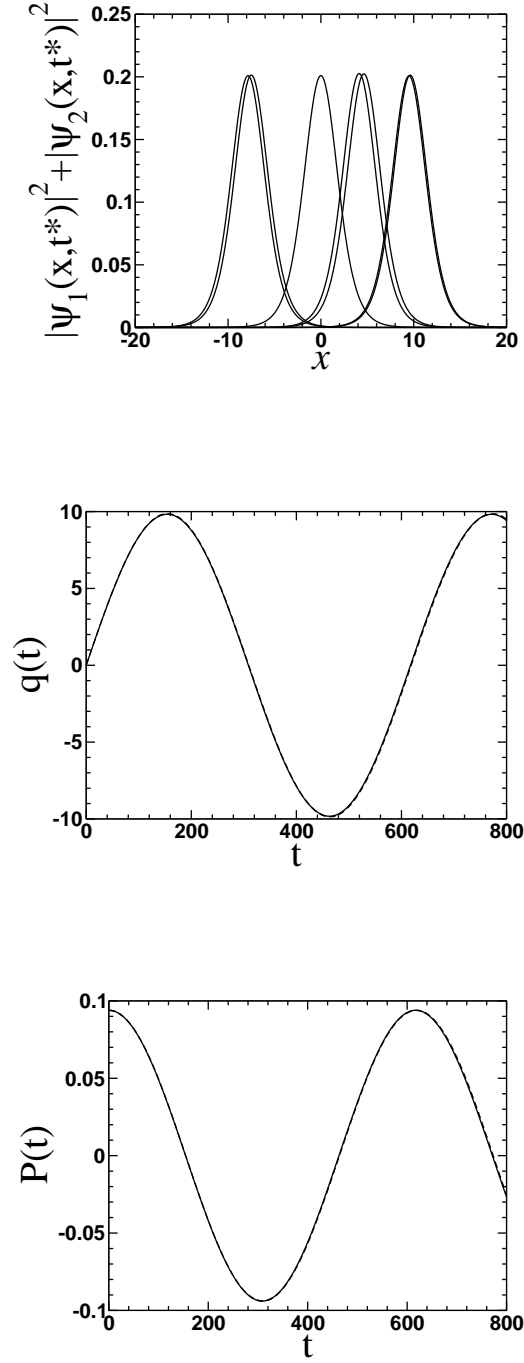


FIG. 11: Harmonic potential, $V(x) = (V_2/2)x^2$. Upper panel: Charge density at $t^* = 0; 133.3; 266.6; 400; 533.3; 666.6; 800$. Middle and lower panels: soliton position $q(t)$ and momentum $P(t)$, from the numerical solutions of the CC equations (solid lines) and from numerical simulations (dashed lines) of the forced NLDE. The curves are super-imposed. The charge $Q = 0.96864$ and energy $E = 0.93921$ are both conserved. Parameters: $g = 1$, $m = 1$, $\omega = 0.9$ and $V_2 = 10^{-4}$. Initial condition: exact soliton of the unperturbed NLDE with initial velocity $v(0) = 0.1$.

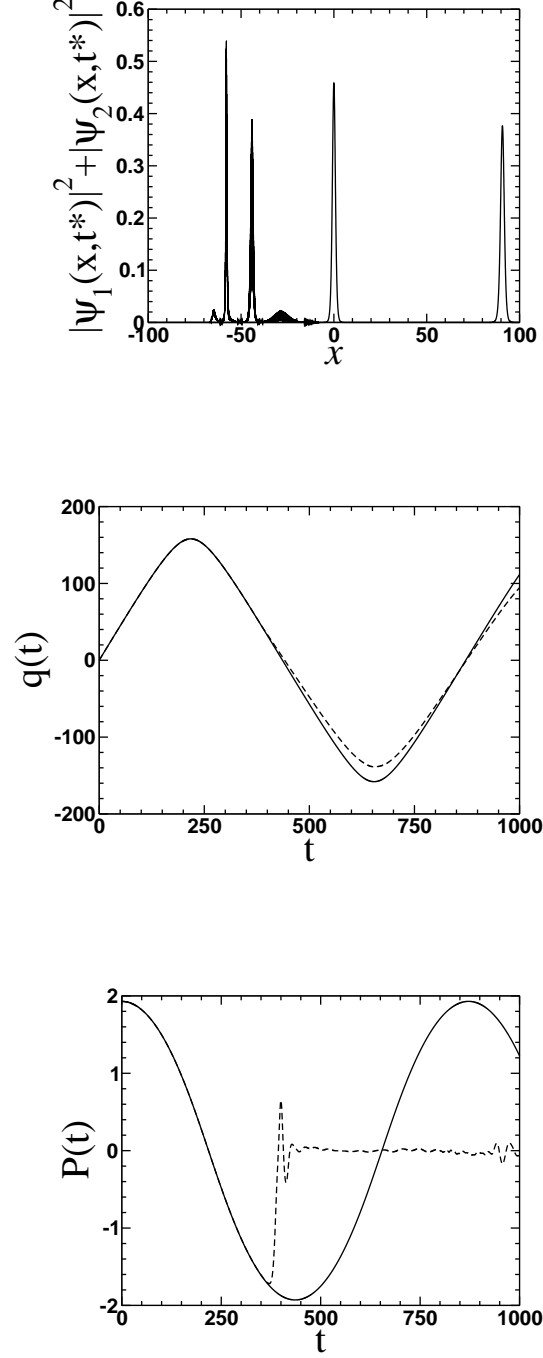


FIG. 12: Harmonic potential, $V(x) = (V_2/2)x^2$. Relativistic regime, unstable soliton. Upper panel: Charge density at $t^* = 0; 133.3; 500$. Middle and lower panels: soliton position $q(t)$ and momentum $P(t)$, from the numerical solutions of the CC equations (solid lines) and from numerical simulations (dashed lines) of the forced NLDE. The curves in $P(t)$ are super-imposed only till $t = 370$. Parameters: $g = 1$, $m = 1$, $\omega = 0.9$ and $V_2 = 10^{-4}$. Initial condition: exact soliton of the unperturbed NLDE with initial velocity $v(0) = 0.9$.

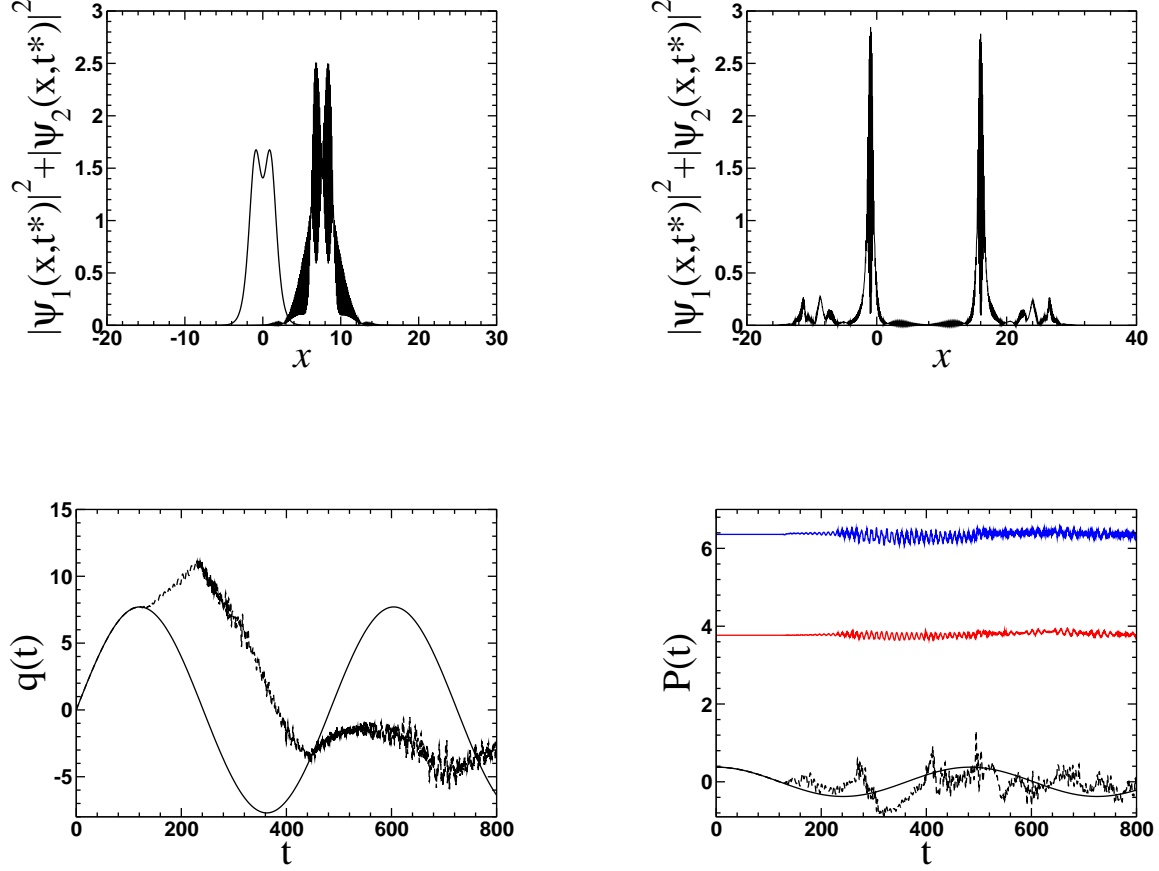


FIG. 13: (Color online). Harmonic potential, $V(x) = (V_2/2)x^2$ with ω in the unstable regime. Left upper panel: Charge density at $t^* = 0;133.3$. Right upper panel: Charge density at $t^* = 150$. Lower panels: soliton position $q(t)$ and momentum $P(t)$, from the numerical solutions of the CC equations (black solid lines) and from numerical simulations (dashed lines) of the forced NLDE. The energy (red or middle curve) and charge (blue or upper curve) are also plotted. Parameters: $g = 1$, $m = 1$, $\omega = 0.3$ and $V_2 = 10^{-4}$. Initial condition: exact soliton of the unperturbed NLDE with initial velocity $v(0) = 0.1$.

D. Spatially periodic potentials

Next consider a spatially periodic potential

$$V(x) = -\epsilon \cos kx, \quad \epsilon > 0, \quad (7.25)$$

where the spatial period $L = 2\pi/k \gg 1/\beta$, and $1/\beta$ is the width b of the solitary wave. The potential is then a function of q, \dot{q} and is given by

$$U[q, \dot{q}] = -\epsilon \cos kq \int dz \cos \frac{kz}{\gamma} [A^2[z] + B^2[z]] = -\epsilon \cos kq I_4[\dot{q}], \quad (7.26)$$

so that

$$\frac{\partial U}{\partial q} = k\epsilon \sin kq I_4[\dot{q}], \quad (7.27)$$

$$\frac{\partial U}{\partial \dot{q}} = -k\epsilon \gamma \dot{q} \cos kq \int dz z \sin \frac{kz}{\gamma} [A^2[z] + B^2[z]] \equiv -k\epsilon \gamma \dot{q} \cos kq I_5[\dot{q}]. \quad (7.28)$$

The generalized force is obtained from

$$F_{eff}[q, \dot{q}] = \frac{d}{dt} \frac{\partial U}{\partial \dot{q}} - \frac{\partial U}{\partial q}. \quad (7.29)$$

We find that

$$\frac{d}{dt} \frac{\partial U}{\partial \dot{q}} = -k\epsilon \cos kq \ddot{q} [\gamma^3 I_5[\dot{q}] - k(\gamma \dot{q})^2 I_6[\dot{q}]], \quad (7.30)$$

where

$$I_6[\dot{q}] = \int dz \, z^2 \cos \frac{kz}{\gamma} [A^2[z] + B^2[z]]. \quad (7.31)$$

The generalized force equation can be written as

$$F[q, \dot{q}] = -k\epsilon I_4(\dot{q}) \sin(kq(t)) - k\epsilon \cos kq \ddot{q} [\gamma^3 I_5[\dot{q}] - k(\gamma \dot{q})^2 I_6[\dot{q}]] = M_0 \frac{d}{dt}(\gamma \dot{q}). \quad (7.32)$$

In the nonrelativistic limit $(\dot{q})^2 \ll 1$, $\gamma \approx 1$ and we obtain for the force law:

$$(M_0 + k\epsilon I_5^0 \cos kq) \ddot{q} + k\epsilon I_4^0 \sin kq = 0, \quad (7.33)$$

where

$$I_4^0 = \int dz \, \cos kz [A^2[z] + B^2[z]]; \quad I_5^0 = \int dz \, z \sin kz [A^2[z] + B^2[z]]. \quad (7.34)$$

When the potential is weak ($\epsilon \ll 1$) then $k\epsilon I_5^0 \ll M_0$ and we obtain the pendulum (or sine-Gordon) equation

$$M_0 \ddot{q} + k\epsilon I_4^0 \sin kq = 0. \quad (7.35)$$

Letting $C = k\epsilon I_4^0/M_0$, the solutions are given by

$$q(t) = \frac{2}{k} \text{am} \left(\frac{1}{2} \sqrt{k} \sqrt{2Ct^2 + kc_1 t^2 + 4Cc_2 t + 2kc_1 c_2 t + 2Cc_2^2 + kc_1 c_2^2}, \frac{4C}{2C + kc_1} \right), \quad (7.36)$$

where c_1, c_2 are integration constants to be determined by the initial conditions, $q(0) = 0; \dot{q}(0) = v_0$. Here

$$\text{am}[u, l] = \text{JacobiAmplitude}[u, l], \quad (7.37)$$

where the modulus parameter l (usually denoted by m) is $l = \frac{4C}{2C + kc_1}$. For the above initial conditions we find:

$$q(t) = \frac{2}{k} \text{am} \left[\frac{kv_0 t}{2}, \frac{4C}{kv_0^2} \right] = q(t) = \frac{2}{k} \sin^{-1} \text{sn} \left[\frac{kv_0 t}{2}, \frac{4C}{kv_0^2} \right]. \quad (7.38)$$

1. Energy conservation

From Eq. (5.38) we have that the soliton energy is given by

$$E = \gamma M_0 - \epsilon \cos kq I_4[\dot{q}] + \epsilon \cos kq \, k\gamma \dot{q}^2 I_5[\dot{q}]. \quad (7.39)$$

In the nonrelativistic limit we obtain

$$E = M_0 - \epsilon \cos kq I_4^0 + \left(\frac{M_0}{2} + \epsilon k I_5^0 \cos kq \right) \dot{q}^2. \quad (7.40)$$

In the case of a weak potential (except of $M_0 \rightarrow 0$ for $\omega \rightarrow 1$)

$$E = \left(1 + \frac{\dot{q}^2}{2} \right) M_0 - \epsilon \cos kq I_4^0. \quad (7.41)$$

2. Soliton momentum and dynamical stability

The soliton momentum is given by Eq. (5.41) and becomes

$$P = \gamma (M_0 + k\epsilon I_5[\dot{q}] \cos kq) \dot{q}. \quad (7.42)$$

In the nonrelativistic regime we obtain

$$P = (M_0 + k\epsilon I_5^0 \cos kq) \dot{q}. \quad (7.43)$$

The necessary condition for stability

$$\frac{dP}{d\dot{q}} = (M_0 + k\epsilon I_5^0 \cos kq) > 0 \quad (7.44)$$

is satisfied except in the regime where $M_0 \rightarrow 0$ which is when $\omega \rightarrow 1$. In that regime the soliton is very broad and the condition $2\pi/k \gg b$ is not fulfilled.

3. Numerical results for $q(t)$ and $P(t)$

For the pendulum equation there is a critical initial velocity at which the coordinate $q(t)$ makes a transition from periodic motion to unbounded motion. This occurs when the modulus parameter $l = 1$. This yields the condition

$$v_c = \sqrt{\frac{4I_4^0\epsilon}{M_0}}. \quad (7.45)$$

Depending on our choice of parameters, for small enough ϵ , v_c will be in the nonrelativistic regime. We choose ω to be in the stability region for the unforced problem (see Sec. III). For $g = 1$, $m = 1$, $\omega_c = 0.697586$. Choosing $\omega = 0.9$, then the width of the solitary wave is $1/(2\beta) = 1.15$. If we choose $k = 0.1$, then the characteristic wave length $2\pi/k = 62.8 \gg 1/(2\beta)$. From Eqs. (2.55) and (2.56) we have that

$$Q = 0.968644; \quad H_1 = 0.0625108 = I_0, \quad (7.46)$$

$$M_0 = H_1 + \omega Q = 0.934291. \quad (7.47)$$

The other constants for this initial condition from Eq. (7.34) are

$$I_4^0 = 0.94632; \quad I_5^0 = 0.433477. \quad (7.48)$$

We have first compared the analytical solution Eq. (7.38) of the pendulum equation with the numerical solution of Eq. (7.33). For $\epsilon < 0.1$ the results are practically identical, for $\epsilon \geq 1$ deviations occur.

Specifically we have chosen the initial condition $q(0) = 0, \dot{q} = v_0$ for the three cases (1) $v_0 \ll v_c \ll 1$ and then v_0 slightly below (2) and above (3) the critical value v_c , namely

$$v_0 = v_c \mp .001. \quad (7.49)$$

Choosing $\epsilon = 0.001$ yields $v_c = 0.0636619$ which is in the nonrelativistic regime, so we expect Eq. (7.38) to hold. In Fig. 14 we show that for $v_0 = 0.01$ the analytic nonrelativistic result and the numerical solution of Eq. (7.33) give the same results as the solution of the NLDE. We also see that the shape of the charge density does not change in time. In Figs. 15 and 16 we show that just below and above the critical velocity, respectively, the analytical result (7.38) agrees with the numerical solution of Eq. (7.33), but both results differ very slightly from the simulation results.

VIII. CONCLUSIONS

We presented analytical and numerical results for the forced NLDE for the scalar-scalar interaction and arbitrary nonlinearity; for numerics we used $\kappa = 1$. For the forcing terms we used simple test potentials such as ramp, harmonic and periodic potentials. We discussed one criterion for the stability of the solitons. We have given two sufficiency

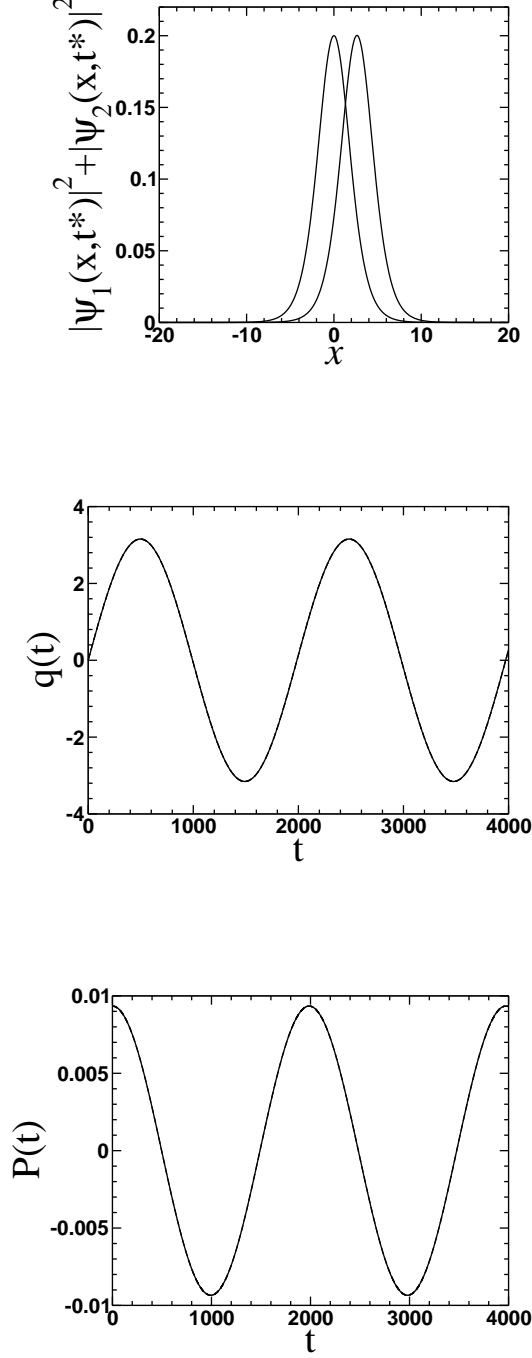


FIG. 14: Periodic potential, $V(x) = -\epsilon \cos(kx)$, very low initial velocity. Upper panel: Charge density at $t^* = 0; 2666.6$. Middle panel: soliton position $q(t)$, from the numerical solution of Eq. (7.33) (solid line), approximate analytical expression (7.38) (dotted line), and from numerical simulations (dashed line) of the forced NLDE. The three curves are super-imposed. Lower panel: momentum $P(t)$, from the numerical solutions of Eq. (7.33) (solid line) and from numerical simulations (dashed line) of the forced NLDE. The curves are super-imposed. Parameters: $g = 1$, $m = 1$, $\omega = 0.9$, $\epsilon = 0.001$ and $k = 0.1$. Initial condition: exact soliton of the unperturbed NLDE with initial velocity $v(0) = 0.01$.

conditions for instability which complement the results given in [14] which are valid only in the nonrelativistic regime where $\omega \leq m$. For one solution the criterion is that the soliton minimizes its energy for fixed charge as a function of ω . Another criterion is a negative slope in dp/dq . Stability was studied numerically for various cases.

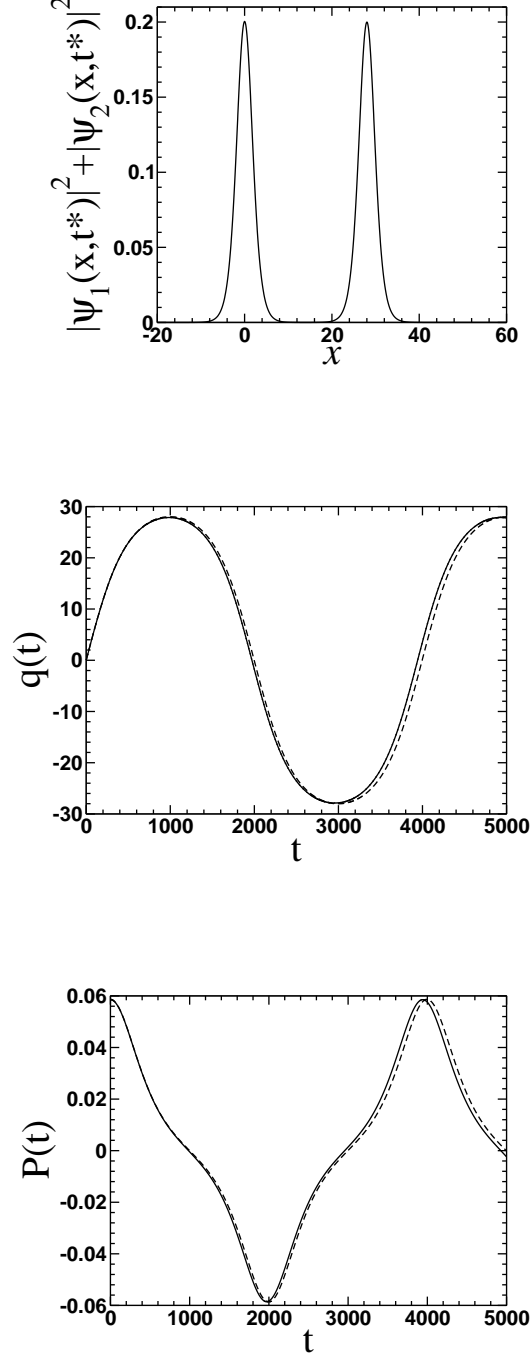


FIG. 15: Spatially periodic potential, $V(x) = -\epsilon \cos(kx)$, initial velocity just below v_c . Upper panel: Charge density at $t^* = 0; 5000$. Middle panel: soliton position $q(t)$, from the numerical solutions of Eq. (7.33) (solid line), approximate analytical expression (7.38) (dotted line), and from numerical simulations (dashed line) of the forced NLDE. Solid and dotted lines are super-imposed. Lower panel: momentum $P(t)$, from the numerical solution of Eq. (7.33) (solid line) and from numerical simulations (dashed line) of the forced NLDE. Parameters: $g = 1$, $m = 1$, $\omega = 0.9$, $\epsilon = 0.001$ and $k = 0.1$. Initial condition: exact soliton of the unperturbed NLDE with initial velocity $v(0) = 0.0626619$.

We also developed a variational ansatz for the NLDE in external fields and compared approximate analytical solutions with numerical solutions of the collective coordinate equations and with simulation results. In all cases, we found very good agreement between the three.

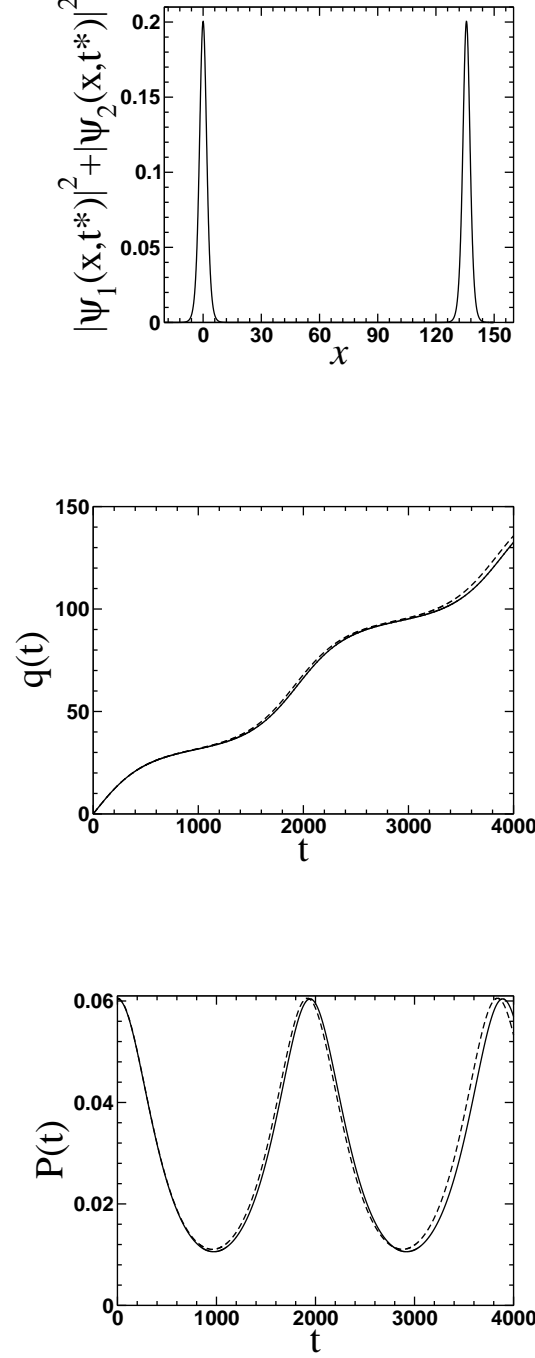


FIG. 16: Spatially periodic potential, $V(x) = -\epsilon \cos(kx)$, initial velocity just above v_c . Upper panel: Charge density at $t^* = 0; 4000$. Middle panel: soliton position $q(t)$, from the numerical solutions of Eq. (7.33) (solid line), approximate analytical expression (7.38) (dotted line), and from numerical simulations (dashed line) of the forced NLDE. Solid and dotted lines are super-imposed. Lower panel: momentum $P(t)$, from the numerical solution of Eq. (7.33) (solid line) and from numerical simulations (dashed line) of the forced NLDE. Parameters: $g = 1$, $m = 1$, $\omega = 0.9$, $\epsilon = 0.001$ and $k = 0.1$. Initial condition: exact soliton of the unperturbed NLDE with initial velocity $v(0) = 0.0646619$.

The simulations in this paper are confined to the $\kappa = 1$ case. The numerical stability of solitons for general κ will be presented in a subsequent publication [17]. The semiclassical reduction of NLDE to NLSE and implications for soliton stability has been recently discussed in a rigorous fashion by Comech [14]. Our numerical findings agree with

his analysis in the non-relativistic regime. We intend to study the forced NLDE with vector-vector interactions and arbitrary nonlinearity in a separate paper.

IX. ACKNOWLEDGMENT

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Appendix: Rest Frame Identities

In the rest frame, energy-momentum conservation leads to identities among the various integrals that arise. The Lagrangian in the axial gauge is given by

$$L = \frac{i}{2} [\bar{\Psi} \gamma^\mu \partial_\mu \Psi - \partial_\mu \bar{\Psi} \gamma^\mu \Psi] - m \Psi + \frac{g^2}{\kappa + 1} (\bar{\Psi} \Psi)^{\kappa+1} - V(x) \Psi^\dagger \Psi. \quad (9.1)$$

In the rest frame, the wave function is given by:

$$\Psi_0 = \psi e^{-i\omega t} = \begin{pmatrix} A(x) \\ i B(x) \end{pmatrix} e^{-i\omega t}. \quad (9.2)$$

The energy-momentum conservation is given by Eq. (2.7) and leads to two independent equations. The first is

$$\partial_0 T^{0x} + \partial_x T^{xx} = 0. \quad (9.3)$$

In the rest frame $T^{0x} = 0$, so that $T^{xx} = \text{constant}$. If the solution goes to zero at infinity then the constant is zero. We have then the relationship:

$$\begin{aligned} T^{xx} &= \frac{i}{2} [\bar{\Psi} \gamma^x \partial^x \Psi - \partial^x \bar{\Psi} \gamma^x \Psi] + L \\ &= \omega \psi^\dagger \psi - m \bar{\psi} \psi + \frac{g^2}{\kappa + 1} (\bar{\psi} \psi)^{\kappa+1} - V(x) \psi^\dagger \psi = 0. \end{aligned} \quad (9.4)$$

Integrating over space we get the relations:

$$\omega Q - m I_1 + \frac{g^2}{\kappa + 1} I_2 - \int dx \rho(x) V(x) = 0. \quad (9.5)$$

The second conservation law is

$$\partial_0 T^{00} + \partial_x T^{x0} = 0, \quad (9.6)$$

which leads to the conservation of energy. The energy of the solitary wave in the rest frame defines the rest mass M_0

$$E = \int T^{00} dx = M_0. \quad (9.7)$$

We have that

$$\begin{aligned} T^{00} &= -\frac{i}{2} [\bar{\psi} \gamma^x \partial_x \psi - \partial_x \bar{\psi} \gamma^x \psi] \\ &\quad + m \bar{\psi} \psi - \frac{g^2}{\kappa + 1} (\bar{\psi} \psi)^{\kappa+1} + V(x) \psi^\dagger \psi \\ &= (AB_x - BA_x) + m(A^2 - B^2) - \frac{g^2}{\kappa + 1} (A^2 - B^2)^{\kappa+1} + (A^2 + B^2)V(x). \end{aligned} \quad (9.8)$$

Integrating we obtain

$$M_0 = I_0 + mI_1 - \frac{g^2}{k+1}I_2 + \int \rho(x)V(x). \quad (9.9)$$

Using the identity of Eq. (9.5), we then have even in the presence of interactions that

$$M_0 = I_0 + \omega Q. \quad (9.10)$$

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- [1] R. J. Finkelstein, C. Fronsdal, and P. Kaus, Phys. Rev. **103**, 1571 (1956); U. Enz, Phys. Rev. **131**, 1392 (1963).
 - [2] M. Soler, Phys. Rev. D **1**, 2766 (1970).
 - [3] W. Strauss and L. Vazquez, Phys. Rev. D **34**, 641 (1986).
 - [4] S.Y. Lee, T. K. Kuo, and A. Gavrielides, Phys. Rev. D **12**, 2249 (1975).
 - [5] Y. Nogami and F. M. Toyama, Phys. Rev. A **45**, 5258 (1992).
 - [6] D. J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974).
 - [7] W. Thirring, Ann. Phys. **3**, 91 (1958).
 - [8] A. Alvarez and B. Carreras, Phys. Lett. **86A**, 327 (1981).
 - [9] F. Cooper, A. Khare, B. Mihaila, and A. Saxena, Phys. Rev. E **82**, 036604 (2010).
 - [10] N.R. Quintero, F.G. Mertens, and A. R. Bishop, Phys. Rev. E **82**, 016606 (2010).
 - [11] F. G. Mertens, N. R. Quintero, and A. R. Bishop, Phys. Rev. E **81**, 016608 (2010).
 - [12] F.G. Mertens, N.R. Quintero, I. Barashenkov, and A. R. Bishop, Phys. Rev. E **84**, 026614 (2011).
 - [13] F. Cooper, A. Khare, N. R. Quintero, F. G. Mertens, and A. Saxena, Phys. Rev. E **85** 046607 (2012).
 - [14] A. Comech, arXiv:1203.3859 and references therein.
 - [15] N.G. Vakhitov and A.A. Kolokolov, Radiophys. Quantum Electron. **16**, 783 (1973).
 - [16] I.L. Bogolubsky, Phys. Lett. A **73**, 87 (1979).
 - [17] N. R. Quintero, F. G. Mertens, F. Cooper, A. Khare, and A. Saxena, “Stability of Solitary waves in the Dirac Equation with arbitrary nonlinearity” (in preparation).